PARALELNOST I NORMALNOST U PROSTORU

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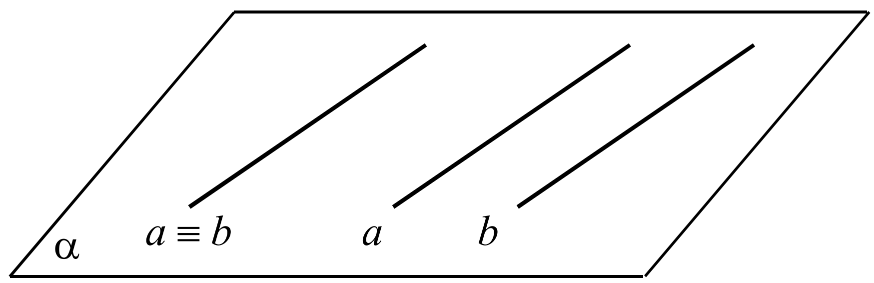
1. PARALELNOST

V_E (Playfair 1860) Za svaku pravu s i svaku tačku O, takvu da $O \notin s$, postoji jedinstvena prava t u ravni sO koja sadrži tačku O i ne seče pravu s.

1.1. Paralelnost pravih

$$a \parallel b \iff a \text{ i } b \text{ komplanarne} \wedge (a \equiv b \vee a \cap b = \varnothing)$$



TEOREMA 1.1.1. *Za svaku pravu s i svaku tačku O u prostoru, postoji jedna i samo jedna prava t takva da $O \in t$ i $t \parallel s$.* ∎

TEOREMA 1.1.2. *U prostoru su date tri nekomplanarne prave od kojih su svake dve komplanarne. Tada važe sledeća tvrđenja:*

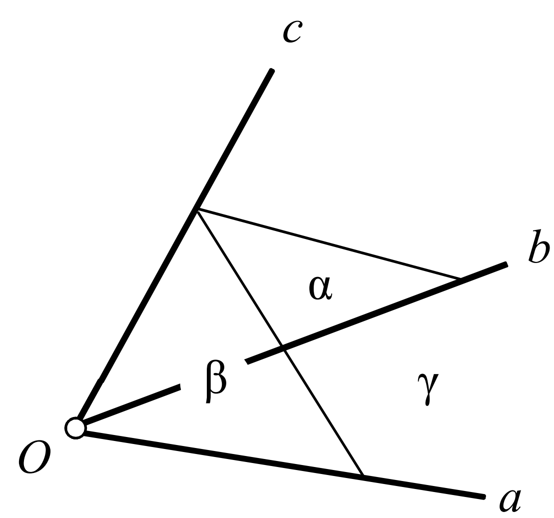
(a) ako se dve prave seku, tada i treća prolazi kroz tačku preseka;

(b) ako su dve prave paralelne, tada je i treća paralelna sa svakom od njih.

Dokaz. $\alpha(b, c)$, $\beta(c, a)$, $\gamma(a, b)$

(a) $a \cap b = \{O\}$

$$\left.\begin{array}{l} O \in a \;\Rightarrow\; O \in \beta \\ O \in b \;\Rightarrow\; O \in \alpha \end{array}\right\} \Rightarrow O \in \alpha \cap \beta = c$$



(b) $a \parallel b$

pretp. $c \nparallel a$

$$\left.\begin{array}{l} \Rightarrow c \cap a = \{O_1\} \\ \overset{(a)}{\Rightarrow} O_1 \in b \end{array}\right\} \lightning\; a \parallel b$$

■

TEOREMA 1.1.3. *Neka su a, b, c tri prave u prostoru. Ako je $a \parallel b$ i $b \parallel c$, tada je $a \parallel c$.*

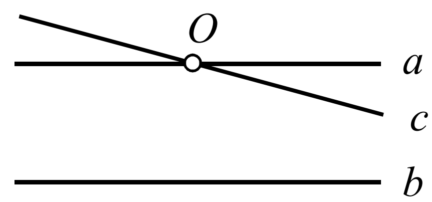
Dokaz. 1° $a = b \vee b = c \vee c = a \Rightarrow a \parallel c$ (trivijalno)

2° $a \neq b \wedge b \neq c \wedge c \neq a$

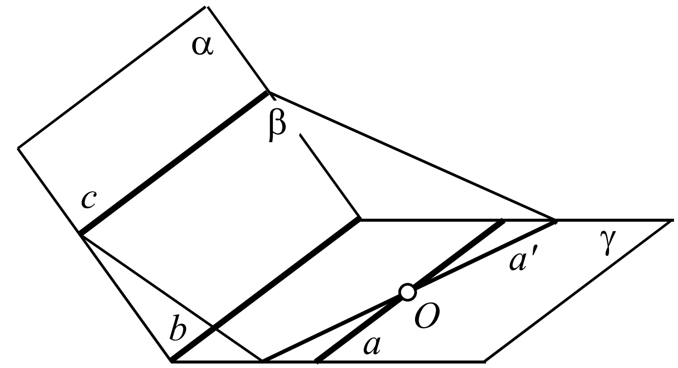
(a) a, b, c – komplanarne

pretp. $a \nparallel c$

$a \cap c = \{O\}$ ↯ V_E



(*b*) a, b, c – nekomplanarne



$a \parallel b \qquad \gamma(a, b)$

$b \parallel c \qquad \alpha(b, c)$

$O \in a \qquad \beta(c, O)$

$\beta \cap \gamma = a'\,(O)$

$$a', b, c \quad b \parallel c \quad \overset{\text{T 1.1.2}}{\Rightarrow} \quad a' \parallel b \ \wedge \ a' \parallel c \qquad (1)$$

$$a(O) \parallel b \ \wedge \ a'(O) \parallel b \ \overset{V_E}{\Rightarrow} \ a \equiv a' \qquad (2)$$

$$(1), (2) \ \Rightarrow \ a \parallel c$$

■

TEOREMA 1.1.4. *Paralelnost pravih u prostoru je relacija ekvivalencije.*

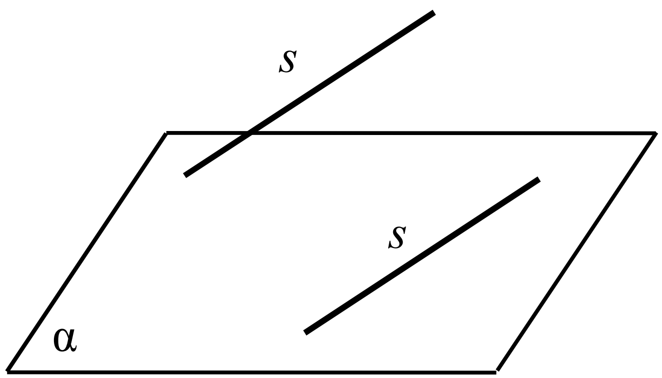
Dokaz. refleksivnost i simetrija iz definicije

tranzitivnost iz T 1.1.3

■

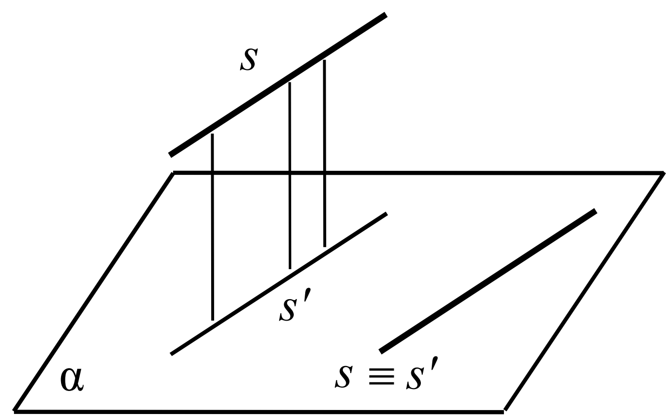
2. Paralelnost pravih i ravni

$s \parallel \alpha \quad \Leftrightarrow \quad s \subset \alpha \ \vee \ s \cap \alpha = \varnothing$



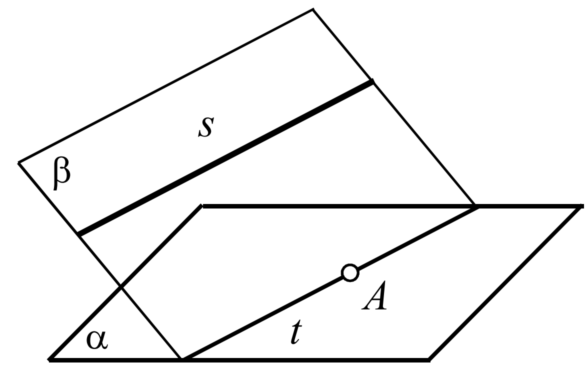
$s \parallel \alpha \quad \Leftrightarrow \quad s \parallel s'$

s' – norm. proj. s na α



TEOREMA 1.2.1. *Prava s je paralelna sa ravni α ako i samo ako postoji prava t u ravni α takva da je $s \parallel t$.*

Dokaz. $(\Rightarrow)$ $s \parallel \alpha$



(a) $s \subset \alpha \qquad s \parallel s \qquad t = s$ ✓

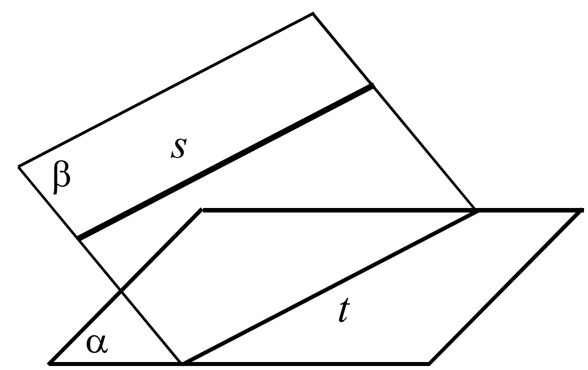
(b) $s \cap \alpha = \varnothing$

$A \in \alpha \quad \beta(s, A) \quad \beta \cap \alpha = t(A) \qquad s \parallel t$

pretp. $s \cap t \neq \varnothing \;\Rightarrow\; s \cap \alpha \neq \varnothing$ ↯ (b)

$\Rightarrow\; s \parallel t$

$(\Leftarrow)$ $s \parallel t\,,\; t \subset \alpha$



(a) $s \subset \alpha \;\Rightarrow\; s \parallel \alpha$ po def. ✓

(b) $s \not\subset \alpha \qquad s \cap \alpha = \varnothing$

pretp. $s \cap \alpha \neq \varnothing$

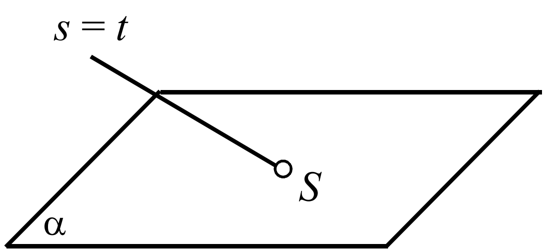
$\beta(s, t) \;\Rightarrow\; s \cap t \neq \varnothing$ ↯ $s \parallel t$

$\Rightarrow\; s \parallel \alpha$

■

TEOREMA 1.2.2. *Ako ravan seče jednu od dve paralelne prave, tada seče i drugu.*

Dokaz. $s \parallel t\,,\ s \cap \alpha = \{S\} \Rightarrow t \cap \alpha \neq \varnothing$



(a) $s = t \qquad t \cap \alpha = s \cap \alpha = \{S\}$ ✓

(b) $s \cap t = \varnothing$

1. varijanta $\quad \beta(s, t) \quad \beta \cap \alpha = a(S)$

LEMA. *Neka su p, q, r tri prave iste ravni pri čemu je p || q. Ako r seče p, tada r seče i q.*

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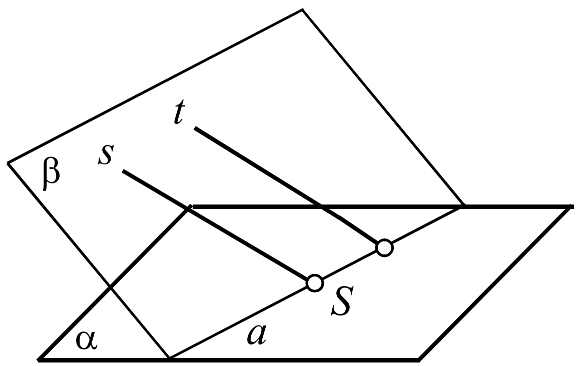
L , $\beta \Rightarrow a$ seče $t \Rightarrow t \cap \alpha \neq \varnothing$



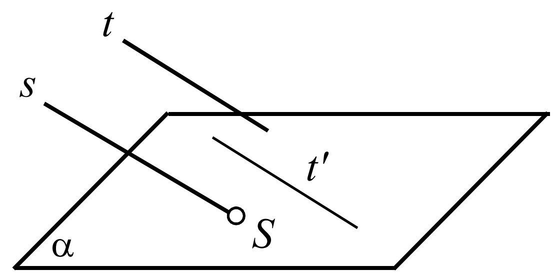
2. varijanta

pretp. $t \cap \alpha = \varnothing \;\Rightarrow\; t \parallel \alpha$

$\text{T } 1.2.1 \;\Rightarrow\; \exists\, t' \subset \alpha\,,\ t' \parallel t$

$s \parallel t \;\wedge\; t \parallel t' \;\overset{\text{T } 1.1.3}{\Longrightarrow}\; s \parallel t'$

$\text{T } 1.2.1 \;\Rightarrow\; s \parallel \alpha \quad ↯ \quad s \cap \alpha = \{S\}$



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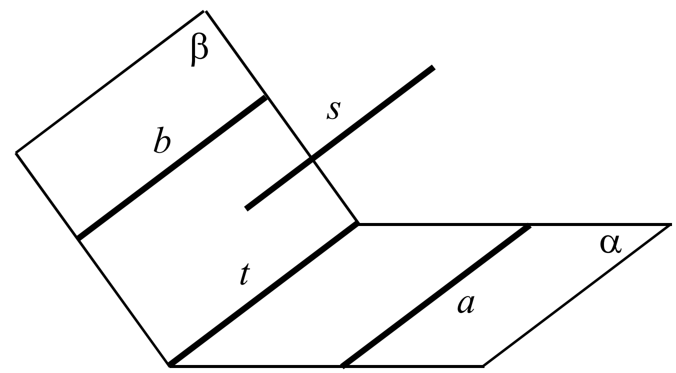
TEOREMA 1.2.3. *Neka je prava s paralelna i sa ravni α i sa ravni β koje se seku po pravoj t. Tada je s || t.*

Dokaz.

$s \parallel \alpha \overset{\text{T 1.2.1}}{\Longrightarrow} \exists\, a \subset \alpha\,,\ s \parallel a$ (1)

$s \parallel \beta \overset{\text{T 1.2.1}}{\Longrightarrow} \exists\, b \subset \beta\,,\ s \parallel b$

$s \parallel a \ \wedge\ s \parallel b \overset{\text{T 1.1.3}}{\Longrightarrow} a \parallel b$ (2)



$a, b, t \overset{\text{T 1.1.1, (2)}}{\Longrightarrow} a \parallel t$ (3)

(1), (3), T 1.1.3 $\Rightarrow\ s \parallel t$

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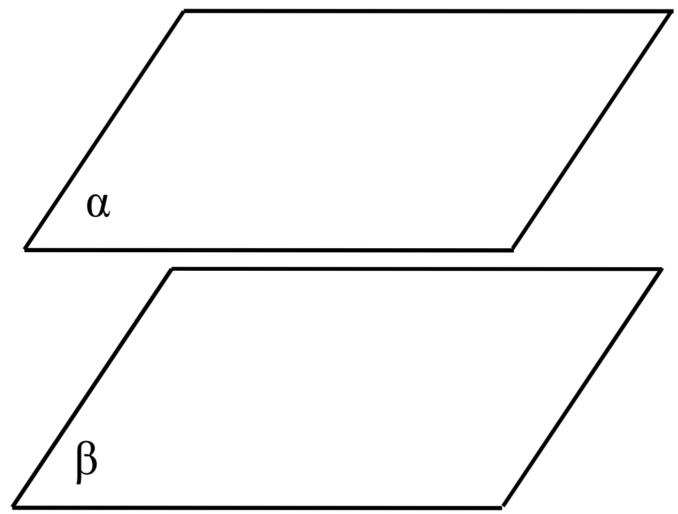
3. Paralelnost ravni

$\alpha \parallel \beta \quad \Leftrightarrow \quad \alpha \equiv \beta \;\vee\; \alpha \cap \beta = \varnothing$



TEOREMA 1.3.1. *Ako je* $\alpha \parallel \beta$ *i* $s \subset \alpha$, *tada je* $s \parallel \beta$.

Dokaz. Iz def. paralelnosti dve ravni i paralelnosti prave i ravni.

■

TEOREMA 1.3.2. *Paralelnost ravni je relacija ekvivalencije.*

Dokaz. refleksivnost i simetrija iz definicije

tranzitivnost $\alpha \parallel \beta \wedge \beta \parallel \gamma \Rightarrow \alpha \parallel \gamma$

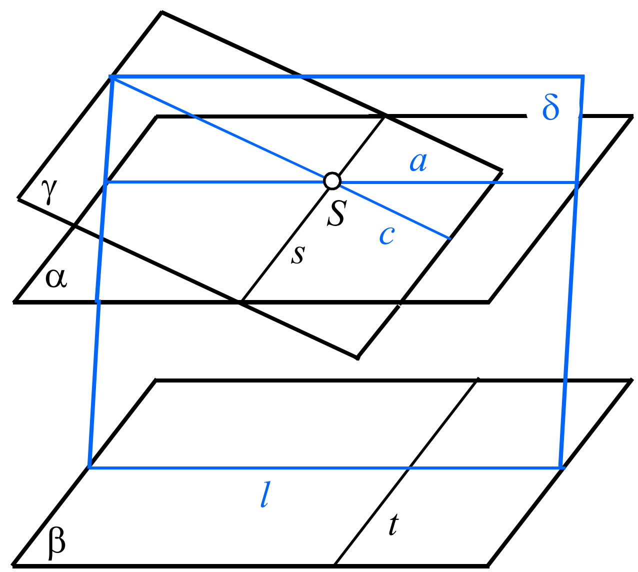
(a) $\alpha = \beta \vee \beta = \gamma \vee \gamma = \alpha \Rightarrow \alpha \parallel \gamma$ (trivijalno)

(b) $\alpha \neq \beta , \beta \neq \gamma , \gamma \neq \alpha$

pretp. $\alpha \cap \gamma = s \overset{\text{T 1.3.1}}{\Rightarrow} s \parallel \beta \overset{\text{T 1.2.1}}{\Rightarrow} \exists\, t \subset \beta ,\ s \parallel t$

$l \subset \beta , l \cap t \neq \varnothing$

$\Rightarrow$ s, l nekomplanarne (mimoilazne)



$S \in s \qquad \delta(l, S)$

$\delta \cap \alpha = a(S) \neq s \qquad \delta \cap \gamma = c(S) \neq s$

$$\left.\begin{aligned} \delta :\ & a(S) \cap l = \varnothing \\ & c(S) \cap l = \varnothing \\ & a \neq c \end{aligned}\right\} \ \lightning\ V_E$$

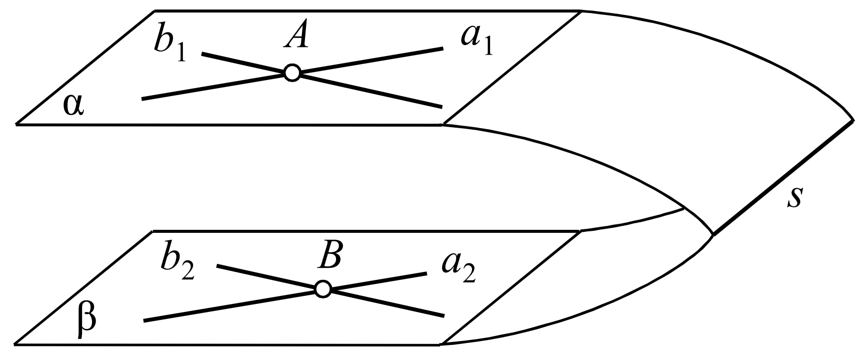
■

TEOREMA 1.3.3. *Neka su* a_1 *i* b_1 *prave ravni* α *koje se seku u tački* A *i neka su* a_2 *i* b_2 *prave ravni* β $(\beta \neq \alpha)$ *koje seku u tački* B. *Ako je* $a_1 \parallel a_2$ *i* $b_1 \parallel b_2$, *tada je* $\alpha \parallel \beta$.

Dokaz. pretp. $\alpha \cap \beta = s$

$$\left.\begin{array}{l} a_1, a_2, s \overset{\text{T 1.1.2}}{\Longrightarrow} a_1(A) \parallel s \\ b_1, b_2, s \overset{\text{T 1.1.2}}{\Longrightarrow} b_1(A) \parallel s \end{array}\right\} ↯\ V_E(\alpha)$$

$\Rightarrow \alpha \parallel \beta$



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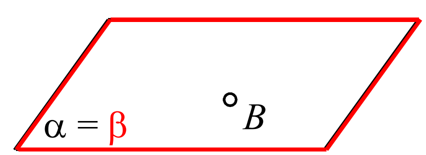
TEOREMA 1.3.4. *Za svaku ravan* α *i svaku tačku* B *postoji jedna i samo jedna ravan* β *takva da* $B \in \beta$ *i* $\beta \parallel \alpha$.

Dokaz.

(a) egzistencija

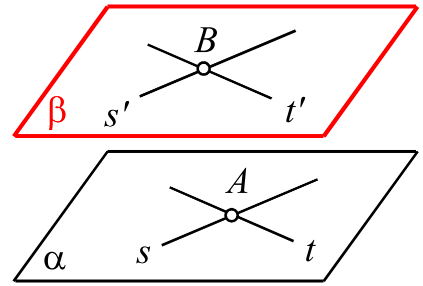
1° $B \in \alpha$ $\quad \beta = \alpha \quad \alpha(B) \parallel \alpha$



$2^{\circ}\ B \notin \alpha \qquad s, t \subset \alpha\ ,\ \ s \cap t = \{A\}$

$s'(B) \parallel s\ ,\ \ t'(B) \parallel t \qquad \beta(s', t')$

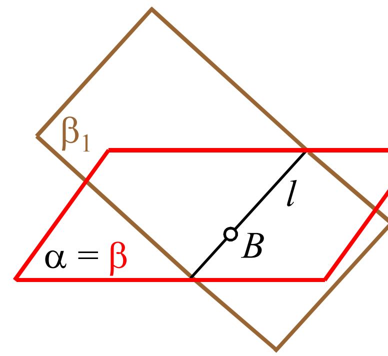
$\text{T } 1.3.3 \ \Rightarrow\ \beta \parallel \alpha$



(b) jedinstvenost

pretp. $\exists\, \beta_1 \neq \beta\ ,\ \ \beta_1(B) \parallel \alpha \qquad (1)$

$\beta_1 \cap \beta = l \qquad (2)$

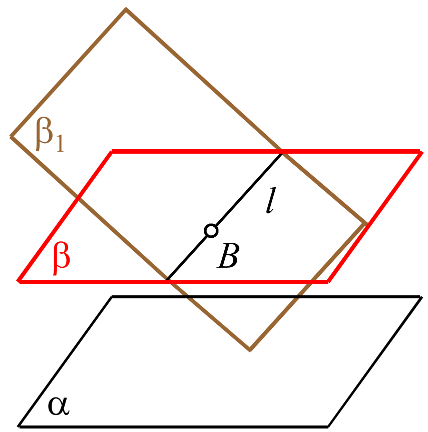


$1^{\circ}\ \ \beta = \alpha$

$\beta_1(B) \nparallel \alpha$ ↯ (1)

$2^{\circ}\ \ \beta \cap \alpha = \varnothing$

$\beta \parallel \alpha \ \wedge\ \beta_1 \parallel \alpha \ \overset{\text{T } 1.3.2}{\Rightarrow}\ \beta \parallel \beta_1$ ↯ (2)



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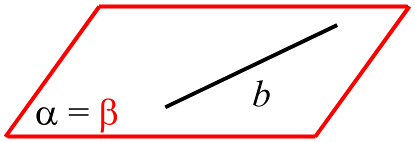
TEOREMA 1.3.5. *Ako je prava b paralelna s ravni α, tada postoji jedna i samo jedna ravan β takva da $b \subset \beta$ i $\beta \parallel \alpha$.*

Dokaz.

(a) egzistencija

1° $b \subset \alpha$ $\quad \beta = \alpha \quad \alpha(b) \parallel \alpha$



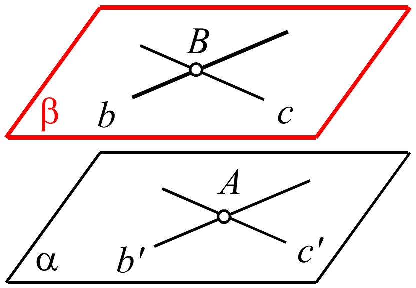
2° $b \not\subset \alpha$

$$b \parallel \alpha \overset{\text{def.}}{\Rightarrow} \exists\, b' \subset \alpha ,\, b \parallel b'$$

$$c' \subset \alpha ,\, c' \cap b' = \{A\}$$

$$B \in b \quad c(B) \parallel c' \quad \beta(b, c)$$

$$\text{T } 1.3.3 \Rightarrow \beta \parallel \alpha$$



(b) jedinstvenost

kao u T 1.3.4

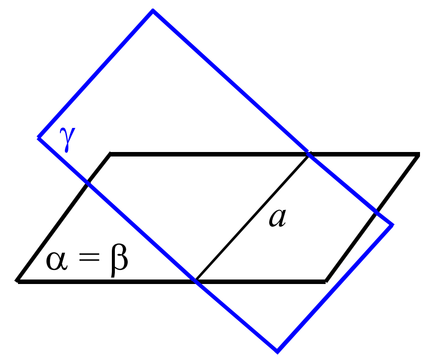
■

TEOREMA 1.3.6. *Ako ravan seče jednu od dve paralelne ravni, tada seče i drugu.*

Dokaz. $\alpha \parallel \beta\,,\ \gamma \cap \alpha = a \ \Rightarrow\ \gamma \cap \beta = b$

(a) $\alpha = \beta$

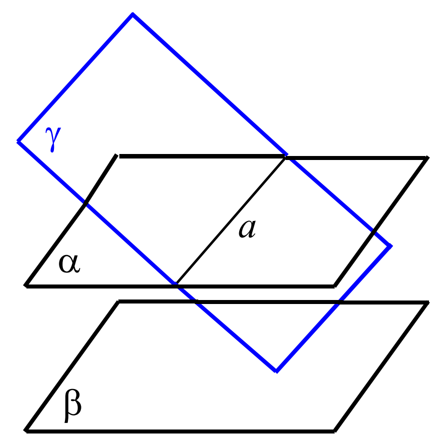
$$\gamma \cap \beta = \gamma \cap \alpha = a \qquad \checkmark$$



(b) $\alpha \cap \beta = \varnothing$

pretp. $\gamma \cap \beta = \varnothing \ \Rightarrow\ \gamma \parallel \beta$

$$\left.\begin{array}{l} \alpha(a) \parallel \beta \\ \gamma(a) \parallel \beta \\ a \parallel \beta \end{array}\right\} \overset{\text{T 1.3.5}}{\Rightarrow} \ \alpha = \gamma \quad \lightning \ \gamma \cap \alpha = a$$



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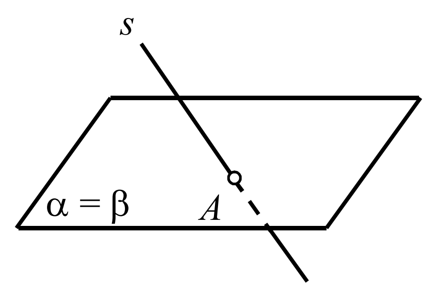
TEOREMA 1.3.6. *Ako prava seče jednu od dve paralelne ravni, tada seče i drugu.*

Dokaz. $\alpha \parallel \beta\,,\ s \cap \alpha = \{A\} \ \Rightarrow\ s \cap \beta \neq \varnothing$

(a) $\alpha = \beta$

$s \cap \beta = s \cap \alpha = \{A\}$ ✓



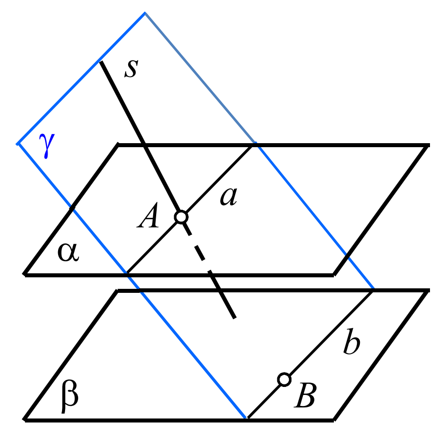
(b) $\alpha \cap \beta = \varnothing$ (1)

pretp. $s \cap \beta = \varnothing$ (2)

$B \in \beta \quad \gamma(s, B)$

$\gamma \cap \alpha = a(A) \quad \gamma \cap \beta = b(B)$

$$\left.\begin{array}{l} (1) \Rightarrow a(A) \parallel b \\ (2) \Rightarrow s(A) \parallel b \end{array}\right\} \lightning \ \mathrm{V}_{\mathrm{E}}(\gamma)$$



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2. NORMALNOST

2.1. Normalnost dve prave

TEOREMA 2.1.1. *Za svaku pravu s i svaku tačka O ravni α, postoji jedna i samo jedna prava t ravni α takva da $O \in t$ i $t \perp s$.*

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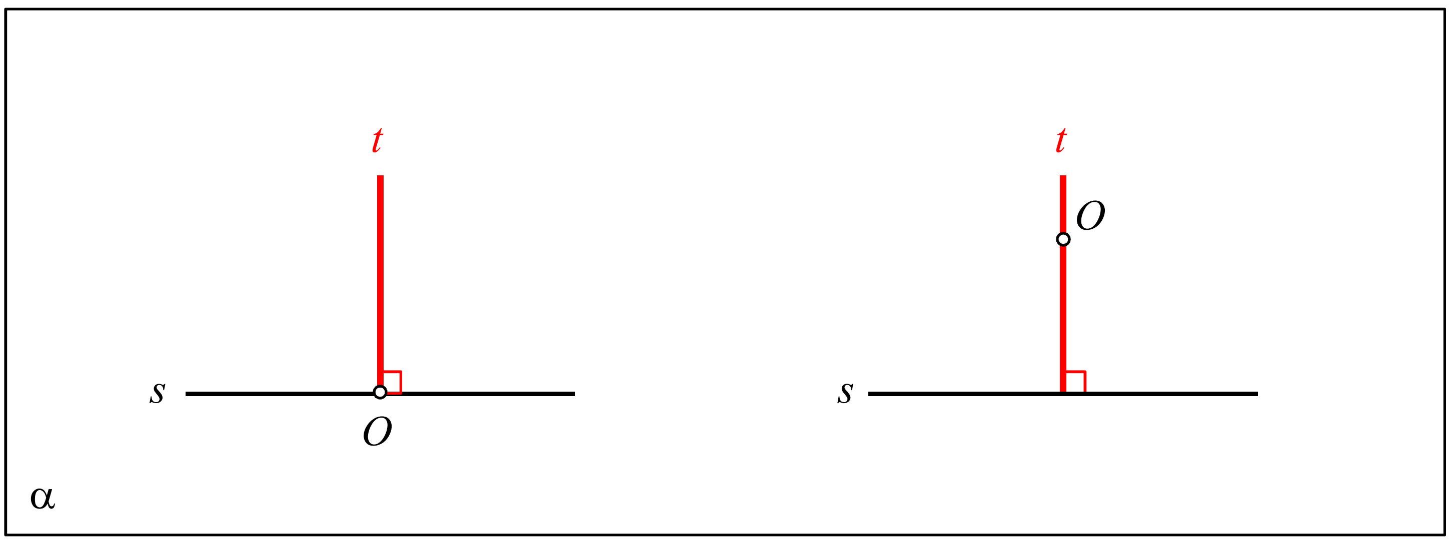
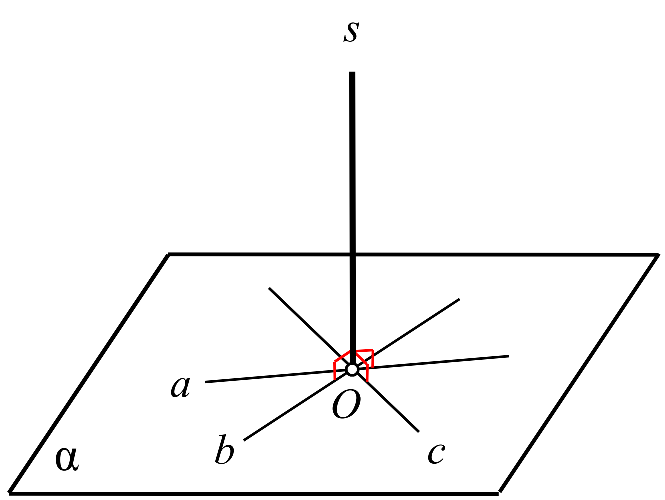


2.2. Normalnost prave i ravni

$s \perp \alpha \quad \Leftrightarrow \quad$ 1. $s \cap \alpha = \{O\}$

2. $\forall\ a(O), b(O), c(O) \ldots \subset \alpha$

$s \perp a, b, c, \ldots$



TEOREMA 2.2.1. *Prava s je normalna na ravan α ako i samo ako je seče i normalna je na dve prave ravni α koje prolaze kroz tačku preseka.*

Dokaz. $s \cap \alpha = \{O\}$

$a\,(O),\, b\,(O) \subset \alpha$

$s \perp a\,,\, s \perp b$

$c\,(O) \subset \alpha \;\Rightarrow\; s \perp c$

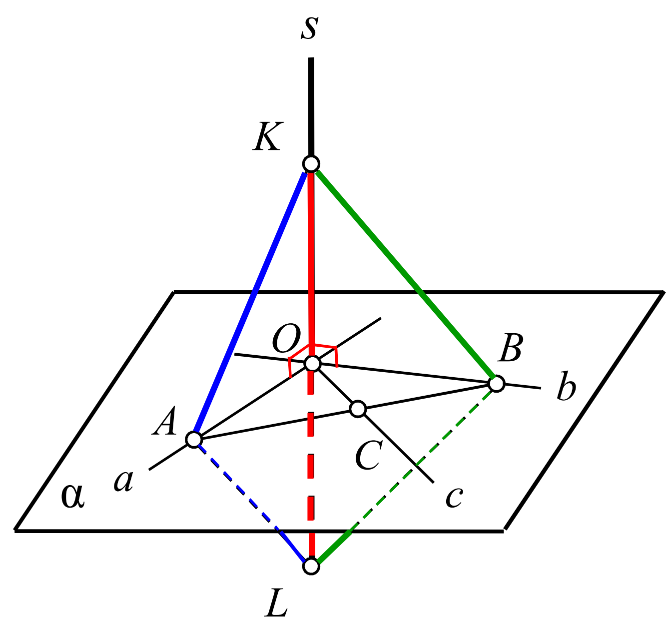
$A \in a\,,\, A \neq O$

$B \in b\,,\, B \neq O$

$c \cap [AB] = \{C\}$

$K \in s\,,\, K \neq O$

$L \in s\; -\; O$ sredina $[KL]$ (*)



$\Delta\, AOK \cong \Delta\, AOL$ (SUS) $\Rightarrow [AK] \cong [AL]$ (1)

$\Delta\, BOK \cong \Delta\, BOL$ (SUS) $\Rightarrow [BK] \cong [BL]$ (2)

$[AK] \cong [AL]$ (1)

$[BK] \cong [BL]$ (2)

$(1), (2) \Rightarrow \Delta ABK \cong \Delta ABL$ (SSS)

$\Rightarrow \angle KAB \cong \angle LAB$ (3)

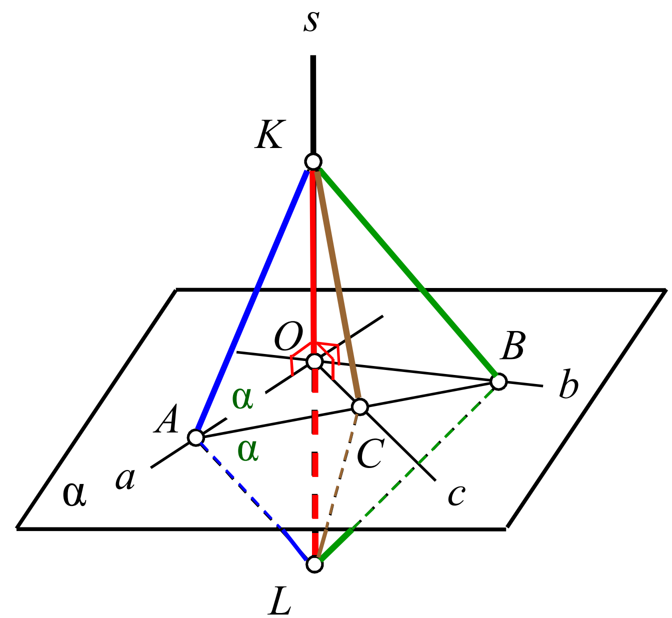
$(1), (3) \Rightarrow \Delta AKC \cong \Delta ALC$ (SUS)

$\Rightarrow [KC] \cong [LC]$ (4)

$(*), (4) \Rightarrow \Delta KOC \cong \Delta LOC$ (SSS)

$\Rightarrow \angle KOC \cong \angle LOC = 90^0$

$\Rightarrow s \perp c$



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TEOREMA 2.2.2. *Za svaku tačku A i svaku pravu s postoji jedna i samo jedna ravan α, takva da $A \in \alpha$ i $\alpha \perp s$.*

Dokaz. 1° $A \in s$

(a) egzistencija

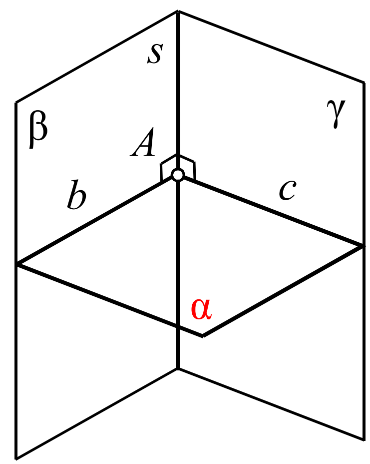
$\beta\,(s)$, $\gamma\,(s)$, $\beta \neq \gamma$

$b\,(A) \subset \beta$, $b \perp s$

$c\,(A) \subset \gamma$, $c \perp s$

$\alpha\,(b, c)$

T 2.2.1 $\Rightarrow$ $s \perp \alpha$



(b) jedinstvenost

pretp. $\exists\ \alpha'\ (A) \perp s\ ,\ \alpha' \neq \alpha$

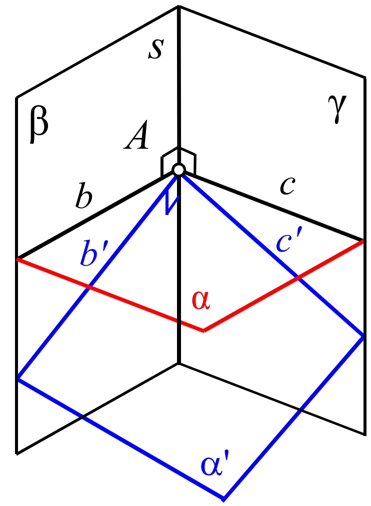
$\alpha' \cap \beta = b'\,(A)$

$\alpha' \cap \gamma = c'\,(A)$

$\alpha' \neq \alpha \;\Rightarrow\; \underline{b' \neq b} \;\vee\; c' \neq c$

$s \perp \alpha' \quad \Rightarrow\; s \perp b'\,(A)$

$$\left.\begin{array}{l} b'\,(A) \perp s \\ b\,(A) \perp s \\ b' \neq b \end{array}\right\} \;\lightning\; \text{T 2.1.1 } (\beta)$$



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2° $A \notin s$

(a) egzistencija

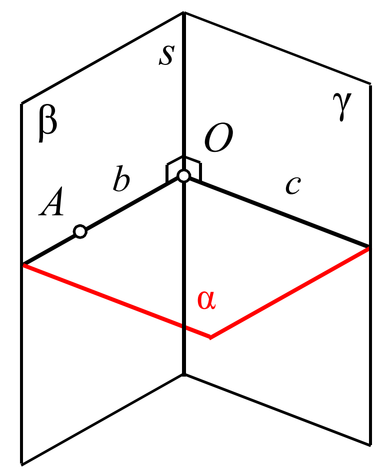
$\beta\,(s, A)$, $\gamma\,(s)$, $\gamma \neq \beta$

$b\,(A) \subset \beta$, $b \perp s$, $b \cap s = \{O\}$

$c\,(O) \subset \gamma$, $c \perp s$

$\alpha\,(b, c)$

T 2.2.1 $\Rightarrow$ $s \perp \alpha$



(b) jedinstvenost

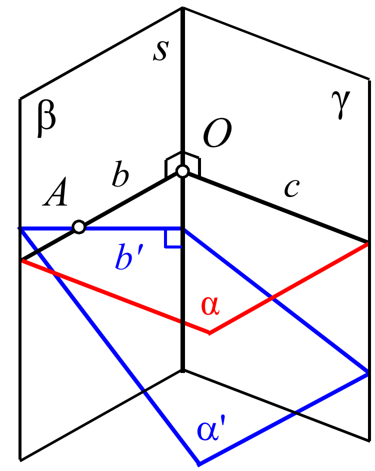
pretp. $\exists\ \alpha'\ (A) \perp s\ ,\ \alpha' \neq \alpha$ (1)

$\alpha' \cap \beta = b'\,(A)$ $\quad b'\,(A) \perp s$

$b' \neq b$ ↯ T 2.1.1 (β)

$\Rightarrow\ b' \equiv b$ (2)

$\Rightarrow\ b' \cap s = \{O\}$

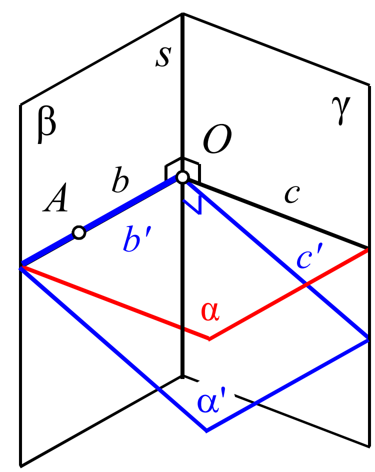


$\alpha' \cap \gamma = c'\,(O)$ $\quad c'(O) \perp s$

$c' \neq c$ ↯ T 2.1.1 (γ)

$\Rightarrow\ c' \equiv c$ (3)

(2), (3) $\Rightarrow\ \alpha' \equiv \alpha$ ↯ (1)



■

TEOREMA 2.2.3. *Za svaku tačku A i svaku ravan* α *postoji jedna i samo jedna prava s, takva da* $A \in s$ *i* $s \perp \alpha$.

Dokaz. 1^o $A \in \alpha$

(a) egzistencija

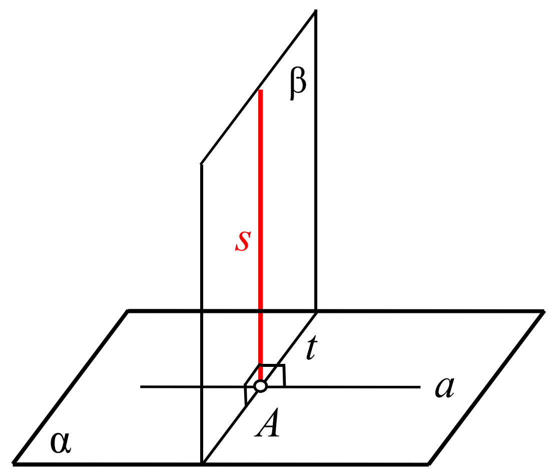
$a(A) \subset \alpha$

T 2.2.2 $\Rightarrow$ $\exists!\, \beta$, $\beta(A) \perp a$

$\beta \cap \alpha = t(A)$

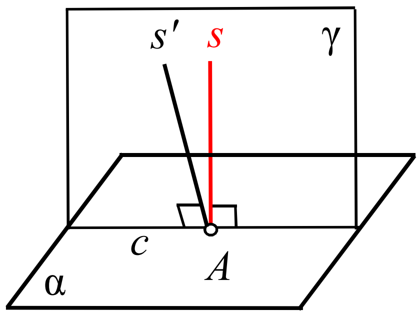
$s(A) \subset \beta$, $s \perp t$ (1)

$a \perp \beta \;\Rightarrow\; s \perp a$ (2)



(1), (2), T 2.2.1 $\Rightarrow$ $s \perp \alpha$

(b) jedinstvenost



pretp. $\exists\, s'$, $s'(A) \perp \alpha$, $s' \neq s$

$s \cap s' = \{A\}$ $\quad\quad \gamma(s, s')$

$\gamma \cap \alpha = c(A)$

$$\left.\begin{array}{l} s(A) \perp \alpha \;\Rightarrow\; s(A) \perp c \\ s'(A) \perp \alpha \;\Rightarrow\; s'(A) \perp c \\ s, s' \subset \gamma,\; s \neq s' \end{array}\right\} \;\lightning\; \text{T 2.1.1 } (\gamma)$$

$2^{\circ}\ A \notin \alpha$

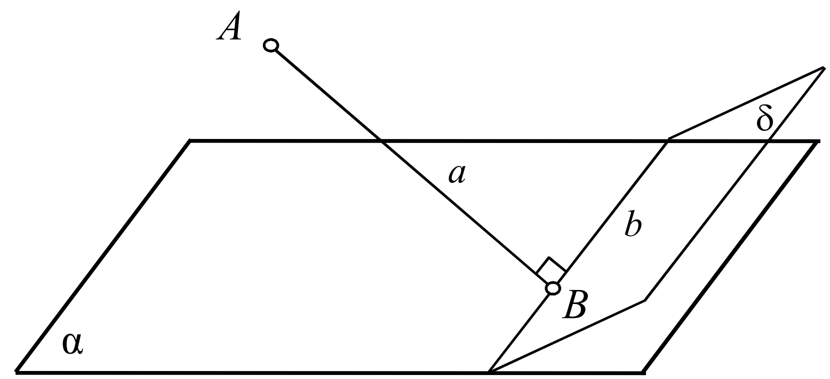
(a) egzistencija

$B \in \alpha$, $a(A, B)$ $\delta(B) \perp a$

$1^{\circ}\ \delta \equiv \alpha \Rightarrow a = s$ ✓

$2^{\circ}\ \delta \neq \alpha \Rightarrow \delta \cap \alpha = b(B)$

$\delta \perp \alpha \Rightarrow b \perp a$



$c(B) \subset \alpha$, $c \perp b$ $\beta(a, c)$

$s(A) \subset \beta$, $s \perp c$ $s \cap c = \{O\}$

$s \perp \alpha$

$C \in b$, $BC \cong AO$

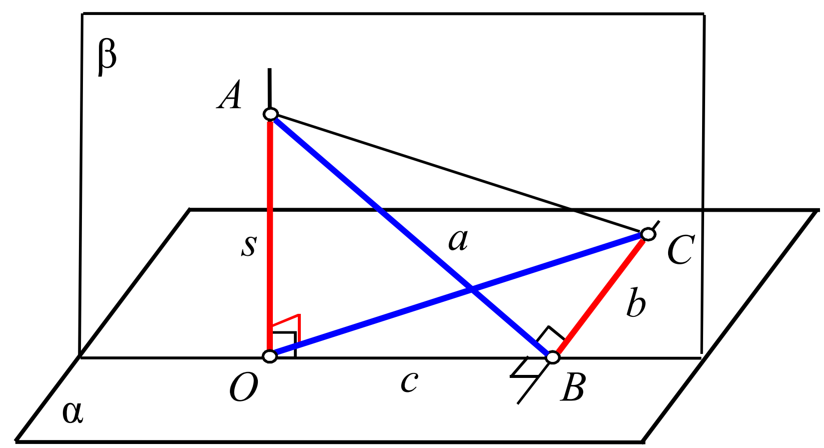
$\Delta OBC \cong \Delta BOA$ (SUS)

$\Rightarrow OC = BA$ (3)

$(3) \Rightarrow \Delta AOC \cong \Delta CBA$ (SSS)

$\Rightarrow \angle AOC = \angle CBA \overset{b \perp a}{=} 90^{\circ}$

$\Rightarrow s \perp OC$

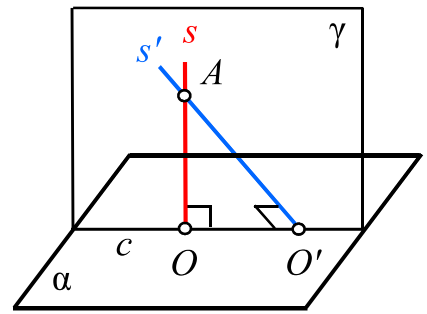


$s \perp OC \wedge s \perp c \overset{\text{T 2.2.1}}{\Rightarrow} s \perp \alpha$ ✓

(b) jedinstvenost

pretp. $\exists\, s',\ s'(A) \perp \alpha\ ,\ s' \neq s$

$s' \cap \alpha = \{O'\}\ ,\ \ O' \neq O$



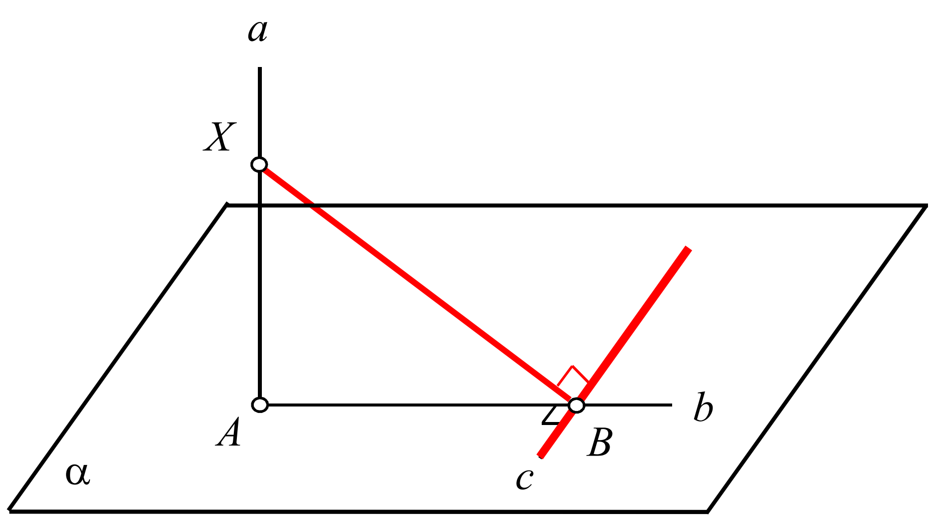
$s \cap s' = \{A\} \qquad \gamma(s, s')$

$\gamma \cap \alpha = c(O, O')$

$$\left.\begin{array}{l} s(A) \perp \alpha \ \Rightarrow\ s(A) \perp c \\ s'(A) \perp \alpha \ \Rightarrow\ s'(A) \perp c \\ s, s' \subset \gamma\ ,\ s \neq s' \end{array}\right\} \ ↯\ \text{T 2.1.1 } (\gamma)$$

$\blacksquare$

TEOREMA 2.2.4.(tri normale) *Prava a je normalna na ravan* α *i seče je u tački A. Neka je b prava ravni* α *koja sadrži tačku A i neka je B tačka prave b,* $B \neq A$*. Neka je c prava ravni* α *koja sadrži tačku B i normalna je na b. Tada je* $BX \perp c$ *za svaku tačku* $X \in a$*.*



Dokaz. I *verzija*

Slično dokazu T 2.2.3 2^{o}

$C \in c$, $BC = AO$ (1)

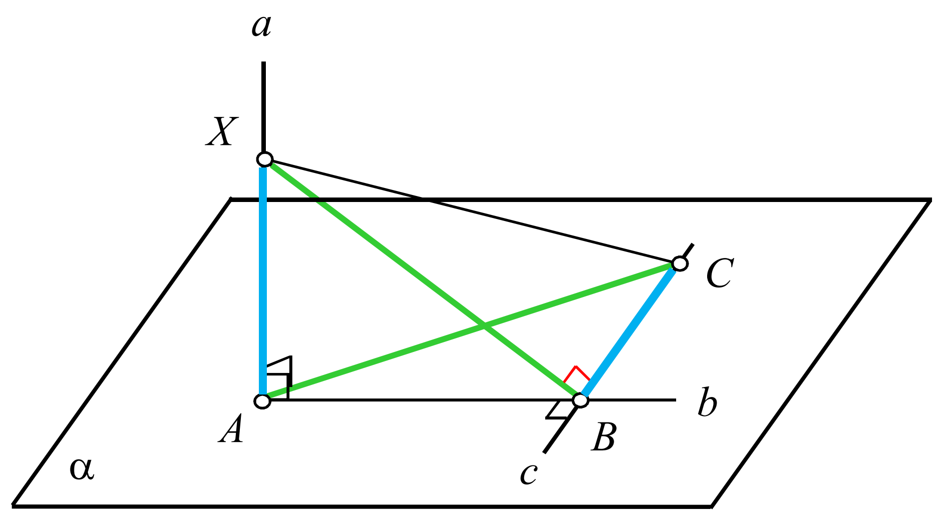
(1) $\Rightarrow$ $\Delta CBA \cong \Delta XAB$ (SUS)

$\Rightarrow$ $CA = XB$ (2)

(2) $\Rightarrow$ $\Delta CBX \cong \Delta XAC$ (SSS)

$\Rightarrow$ $\angle CBX = \angle XAC \overset{a \perp \alpha}{=} 90^{o}$

$\Rightarrow$ $BX \perp c$



II *verzija*

$C \in c\ ,\ C \neq B$

$D \in b\ ,\ C\text{–}B\text{–}D\ ,\ CB = DB$ (3)

(3) $\Rightarrow\ \Delta CBA \cong \Delta DBA$ (SUS)

$\Rightarrow\ CA = DA$ (4)

$a \perp \alpha$

$\Rightarrow\ \angle XAC = \angle XAD = 90^{o}$ (5)

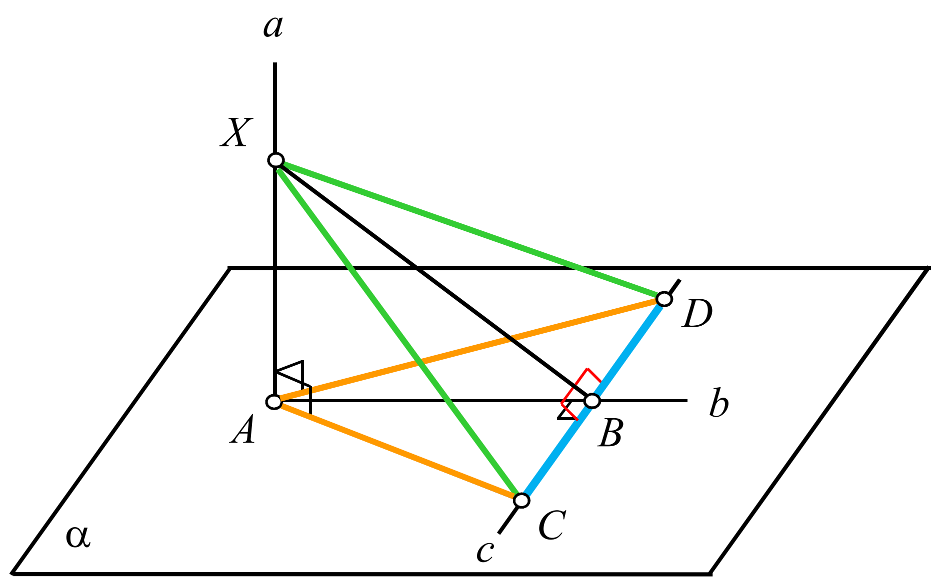
(4), (5) $\Rightarrow\ \Delta XAC \cong \Delta XAD$ (SUS)

$\Rightarrow\ XC = XD$ (6)

(3), (6) $\Rightarrow\ \Delta CBX \cong \Delta DBX$ (SSS)

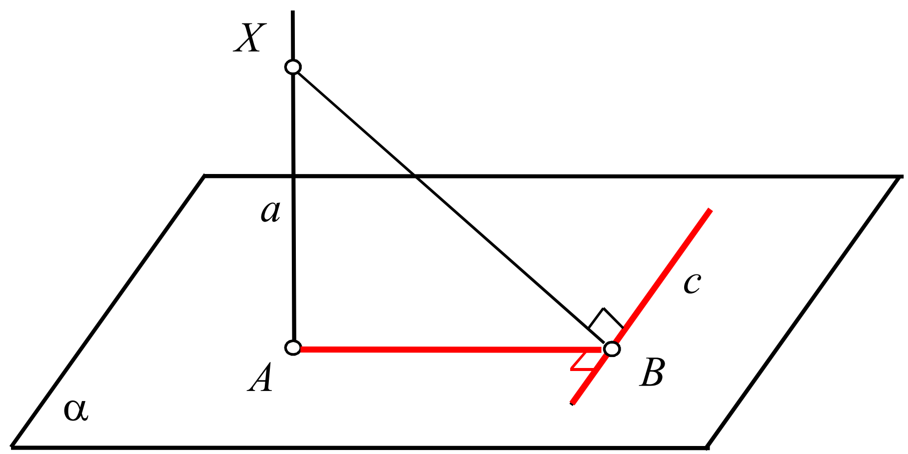
$\Rightarrow\ \angle CBX = \angle DBX = 90^{o}$

$\Rightarrow\ BX \perp c$



■

TEOREMA 2.2.5. (obratna o tri normale) *Prava a je normalna na ravan α i seče je u tački A. Neka je $X \in a$ i neka je B tačka ravni α, $B \neq A$. Neka je c prava ravni α koja sadrži tačku B i normalna je na XB. Tada je $AB \perp c$.*



Dokaz. I *verzija*

Slično dokazu T 2.2.3 2°

$C \in c\ ,\ BC = AX \qquad (1)$

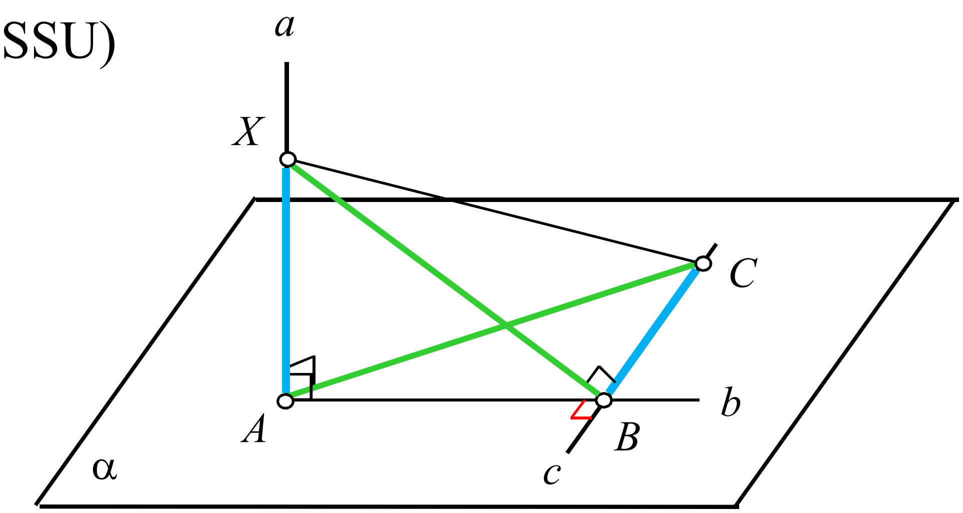
$(1),\ a \perp \alpha \ \Rightarrow\ \Delta CBX \cong \Delta XAC \quad \text{(SSU)}$

$\Rightarrow\ BX = AC \qquad (2)$

$(2)\ \Rightarrow\ \Delta CBA \cong \Delta XAB \quad \text{(SSS)}$

$\Rightarrow\ \angle CBA = \angle XAB \overset{a \perp \alpha}{=} 90^{\circ}$

$\Rightarrow\ AB \perp c$



II *verzija*

$C \in c$, $C \neq B$

$D \in b$, $C–B–D$, $CB = DB$ (3)

(3) $\Rightarrow$ $\Delta CBX \cong \Delta DBX$ (SUS)

$\Rightarrow$ $CX = DX$ (4)

$a \perp \alpha$

$\Rightarrow$ $\angle XAC = \angle XAD = 90^o$ (5)

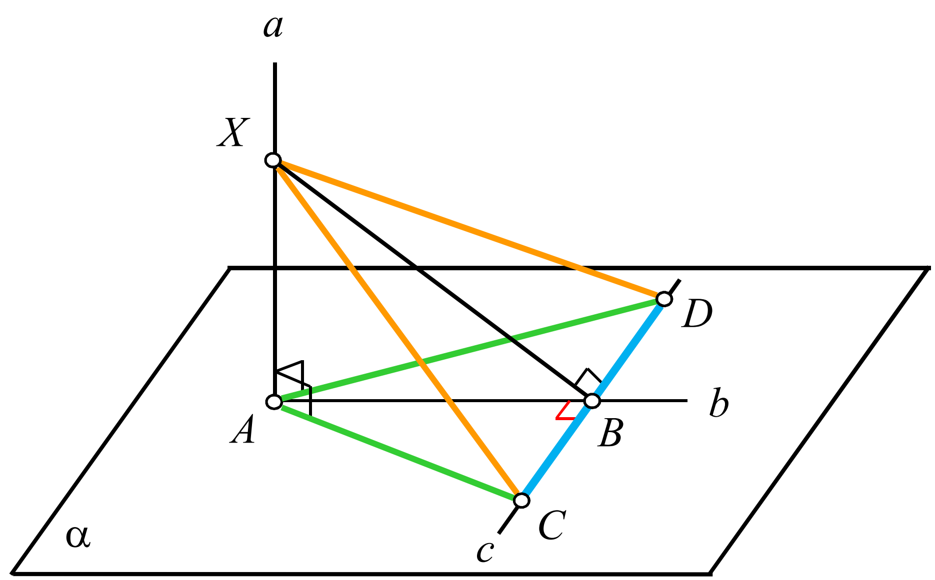
(4), (5) $\Rightarrow$ $\Delta XAC \cong \Delta XAD$ (SSU)

$\Rightarrow$ $AC = AD$ (6)

(3), (6) $\Rightarrow$ $\Delta CBA \cong \Delta DBA$ (SSS)

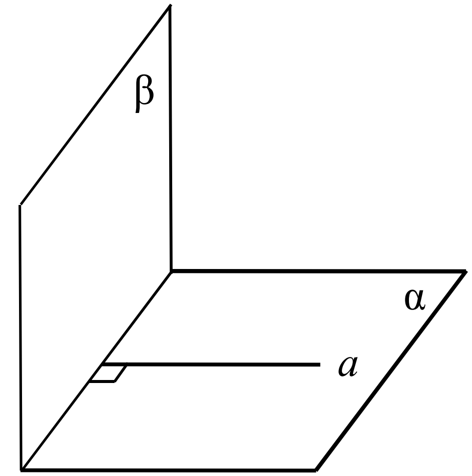
$\Rightarrow$ $\angle CBA = \angle DBA = 90^o$

$\Rightarrow$ $AB \perp c$



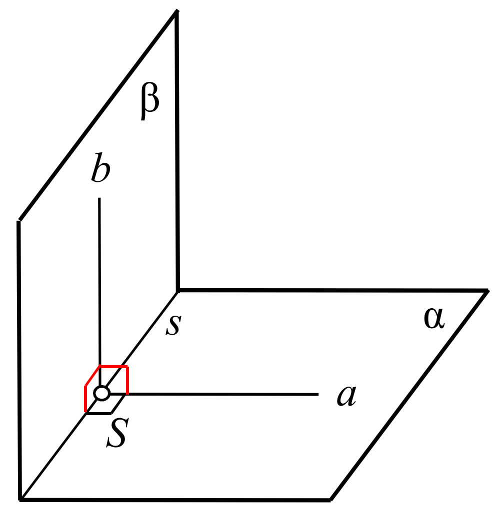
■

2.3. Normalnost dve ravni



$\alpha \perp \beta \iff \exists\, a \subset \alpha\,,\ a \perp \beta$

TEOREMA 2.3.1. *Ako je $\alpha \perp \beta$, tada je i $\beta \perp \alpha$.*



Dokaz. $\alpha \cap \beta = s \qquad a \subset \alpha\,,\ a \perp \beta$

$a \perp s\,,\ a \cap s = \{S\}$

$$\left.\begin{array}{l} b\,(S) \subset \beta\,,\ b \perp s \\ a \perp \beta \ \Rightarrow\ a \perp b \end{array}\right\} \Rightarrow\ b \perp \alpha\ \Rightarrow\ \beta \perp \alpha$$

■

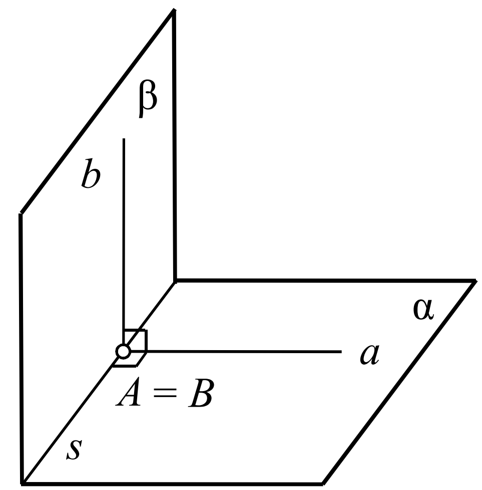
TEOREMA 2.3.2. *Neka je* $\alpha \perp \beta$, $\alpha \cap \beta = s$, $a \subset \alpha$ *i* $a \perp s$. *Tada je* $a \perp \beta$.

Dokaz. $a \cap s = \{A\}$

$$\alpha \perp \beta \overset{\text{T 2.3.1}}{\Rightarrow} \beta \perp \alpha$$

$$\Rightarrow \exists\, b \subset \beta\,,\, b \perp \alpha$$

$$b \cap s = \{B\}$$



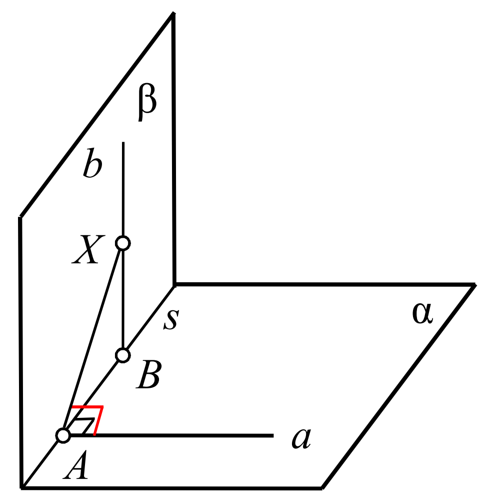
1° $A = B$

$$a \perp s,\ a \perp b \overset{\text{T 2.2.1}}{\Rightarrow} a \perp \beta$$

2° $A \neq B$

$X \in b\,,\ X \neq B$

$$\left.\begin{array}{l} \text{T 2.2.4} \Rightarrow a \perp AX \\ \qquad\qquad\ \ a \perp s \end{array}\right\} \overset{\text{T 2.2.1}}{\Rightarrow} a \perp \beta$$



■

TEOREMA 2.3.3. *Ako je* $\alpha \perp \gamma$ *i* $\beta \perp \gamma$ *i* $\alpha \cap \beta = s$*, tada je* $s \perp \gamma$.

Dokaz. Pretp. $s \not\perp \gamma$ (1)

$\alpha \cap \gamma = a$

$\beta \cap \gamma = b$

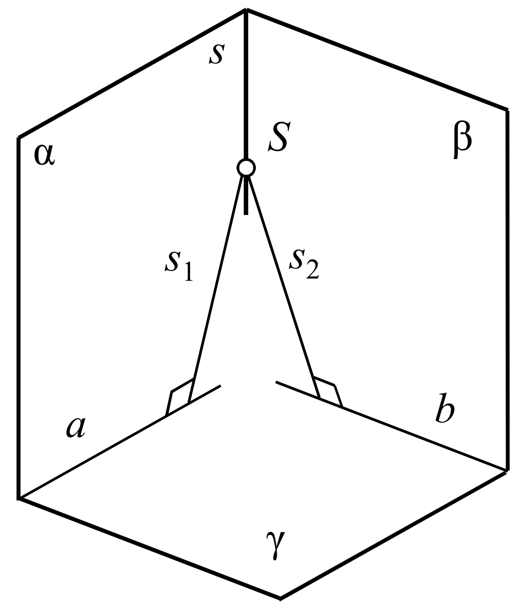
$S \in s$

$s_1(S) \subset \alpha \ , \ s_1 \perp a$

T 2.3.2 $\Rightarrow s_1 \perp \gamma \overset{(1)}{\Rightarrow} s_1 \neq s$

$s_2(S) \subset \beta \ , s_2 \perp b$

T 2.3.2 $\Rightarrow s_2 \perp \gamma \overset{(1)}{\Rightarrow} s_2 \neq s$



$s_1 \neq s_2$

pretp. $s_1 = s_2$

$s_1 \subset \alpha \ \wedge \ s_2(S) \subset \beta \Rightarrow s_1 = s_2 = s \Rightarrow s \perp \gamma$ ↯ (1)

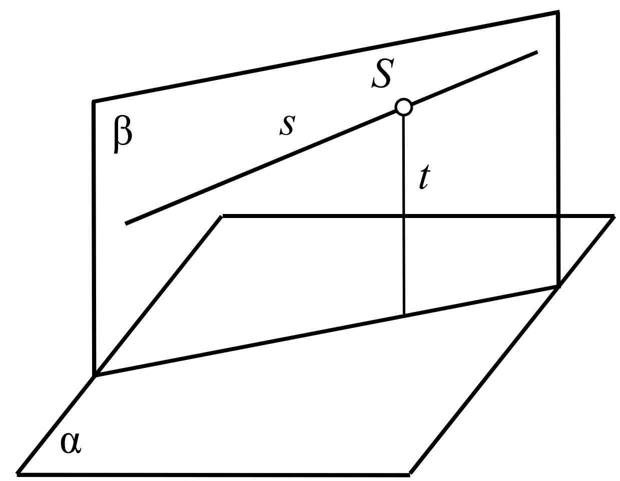
$s_1(S) \perp \gamma \ , s_2(S) \perp \gamma \ , s_1 \neq s_2$ ↯ T 2.2.3

■

TEOREMA 2.3.4. *Ako prava s nije normalna na ravan α, tada postoji jedna i samo jedna ravan β, takva da $s \subset \beta$ i $\beta \perp \alpha$.*

Dokaz.

1° egzistencija



$S \in s$

$T\ 2.2.3 \Rightarrow \exists!\ t\,(S) \perp \alpha$

$s \not\perp \alpha \quad \Rightarrow \quad t \neq s$

$\beta\,(s, t) \perp \alpha$

2° jedinstvenost

pretp. $\exists\ \beta'\,(s) \perp \alpha\ ,\ \beta' \neq \beta$

$\beta' \cap \beta = s$

$T\ 2.3.3 \Rightarrow s \perp \alpha \quad \lightning\ s \not\perp \alpha$

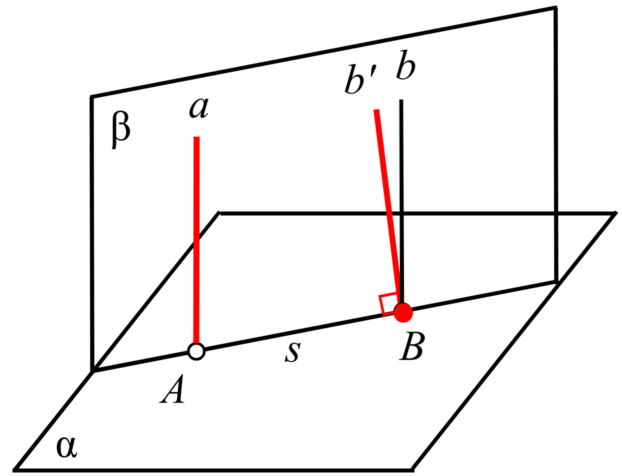
$\blacksquare$

TEOREMA 2.3.5. *Dve prave normale na istu ravan su paralelne.*

Dokaz.

$a \perp \alpha$, $a \cap \alpha = \{A\}$ $\quad$ $b \perp \alpha$, $b \cap \alpha = \{B\}$

1° a i b – komplanarne



pretp. a i b – nekomplanarne

$\beta\,(a, B) \;\Rightarrow\; b \not\subset \beta \qquad \beta \cap \alpha = s\,(A, B)$

$b'\,(B) \subset \beta$, $b' \perp s$

$b' \neq b$

T 2.3.2 $\Rightarrow$ $b' \perp \alpha$

$$\left.\begin{array}{l} b(B) \perp \alpha \\ b'(B) \perp \alpha \\ b \neq b' \end{array}\right\} \;\text{↯}\; \text{T 2.2.3}$$

$\Rightarrow$ a i b – komplanarne $\qquad a, b \subset \beta$

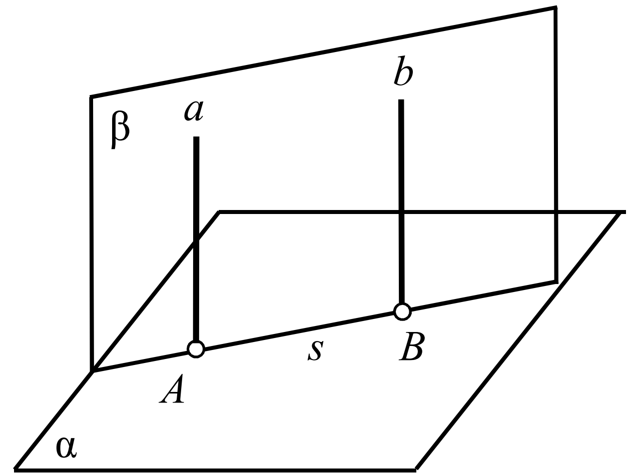
2^{o} $a \cap b = \varnothing$

$a(A) \perp \alpha \qquad b(B) \perp \alpha$

$a, b \subset \beta$, $a \neq b$

pretp. $a \cap b = \{O\}$

$$\left.\begin{array}{l} a(O) \perp \alpha \\ b(O) \perp \alpha \\ a \neq b \end{array}\right\} \lightning \text{ T 2.2.3}$$



$1^{o}, 2^{o} \Rightarrow a \parallel b$

■