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“Research in Mathematics Education”

Proceedings

Edited by Jasmina Milinković and Zoran Kadelburg

Mathematical Society of Serbia

Belgrade, Serbia

May 10 – 11, 2019

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Since all papers are written in English which is, for most authors, not their first language, the responsibility for spelling and grammar lies with the authors of the papers themselves.

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Preview

These proceedings represent a selection of papers presented at the scientific conference held in Belgrade, Serbia, May 10 – 11, 2019. The conference was organized upon the initiative of ERME to support the development of the research community in Eastern Europe. In an effort to follow the international trend of intensification of research in the field of mathematical education, the Mathematical Society of Serbia took the initiative to organize a scientific conference devoted to this issue. The major aim was to introduce and exchange ideas and results of research in mathematics education among math educators in Serbia and neighbouring countries. The themes of plenary presentations traced directions of the research and presented selected findings and perspectives on the proposed topic fields: 1) place and role of different participants in the research in mathematics education, 2) how to use lessons from history and research in educational practice, 3) research methods in mathematics education. There were 41 participants, 3 plenary lectures and 15 reports. The full program is included in these proceedings (p. 134). The proceedings contain two plenary lectures and 9 papers from the authors who decided to send contributions for the proceedings and passed the peer-reviewing process.

We are grateful to ERME and Ministry of Education, Science and Technological Development of Serbia for offering financial support, plenary speakers, contributors, and all participants for taking an active part in the conference, and colleagues who took part in the reviewing process.

Editors

Belgrade, December 2019

Content

Bridging two worlds – Cooperation between academics and teacher-researchers

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The aim of the paper is to present several models of cooperation between mathematics teacher-researchers and academics. Examples of successful teachers' research and collaboration of teachers and academics are presented. Ways in which academics and teacher-researchers might conduct research together serves as the basis for the analysis of factors contributing to the success of these research events. The differences between the roles of teacher-researchers and academics on the one hand and the advantages of the cooperation of both on the other hand are analysed and discussed.

Keywords: Cooperation between academics and teacher-researchers, school and research practice of teachers, communities of practice.

Introduction

The issue of teacher research and cooperation between teachers and academics in research teams in mathematics education is a broad and live topic. In most cases, the focus of these research projects is on the improvement of the quality of mathematics teaching and learning, see e.g. (Brown & Coles, 2000). In many cases, the impact of this type of projects on mathematics education has been analysed – see e.g. (Goos, 2008). The different experience and knowledge proved to be an influencing factor in the findings, e.g. (Brown & Coles; Hošpesová, Macháčková, & Tichá; Lebethe, Eddy, & Bennie; Novotná & Pelantová; Poirier; Rosen; Zack & Reid; in Novotná et al., 2006). Breen (2003, p. 523) presents the contrasting views on the contributions that teachers make to the field of mathematics education: "… there is a growing movement for more teachers to become involved in a critical exploration of their practice through such methods as critical reflection, action research, and lesson studies. The contrasting position makes the claim that these activities have done little to add to the body of knowledge on mathematics education."

Theoretical background

Differences between school and research practice of teachers

The differences and similarities in the school and research practices were described in details by Brousseau (2002, e-mail communication): "When I am a *didactician*, the interpretation of every step of teaching begins with a systematic infirming, a complex work of the analysis a priori and the confrontation with various aspects of contingency, of observations viewed and rejected later etc.

There is not an evident separation of what is relevant but inadequate, adequate but inadaptable, eligible but inconsistent, as well as transformations of appearance and certainties in falsifiable questions etc. When I am a *teacher*, I have to take a number of instantaneous decisions in every moment based on the real information got in the same moment. I can use only very few of subtle conclusions of my work as didactician and I have to fight with starting to pose myself questions which are not compatible with the time that I have and that finally have the chance to be inappropriate for the given moment. I react with my experience, with my knowledge of my pupils, with my knowledge of a teacher of mathematics, which I am treating. All these things are not to be known by the didactician …"

In the section *Teachers' and academics' research*, differences between the roles of teachers and researchers are illustrated on an episode from action research conducted by a Czech teacher.

Communities of practice

Without any doubt cooperation of teacher-researchers and university researchers in the so-called communities of practice enriches the findings about mathematics education (Jaworski, 2001). Wagner (1997) describes three different forms of direct researcher-practitioner cooperation (data-extraction agreements, clinical partnership, co-learning agreements), each of which has different implications on educational research. Jaworski (2001) adopts Wagner's (1997) idea of co-learning between researchers and practitioners in educational settings. She claims that the complementary roles of teachers and academics in their cooperation are significant. She recognises the value of teachers as well as academics engaged in common effort to improve mathematics teaching.

Biza, Jaworski and Hemmi (2014) adopted sociocultural theory and community of practice theory for investigating communities of practice (CoP) in university mathematics environments. They describe learning as "a process of participation and reification in a community in which individuals belong and form their identity through engagement, imagination and alignment". For the cases when inquiry is considered as a fundamental mode of participation, the term community of inquiry (CoI) is used. Proof and conceptual understanding of mathematics at the university level are used for illustrating the used terms. The authors conclude with a critical reflection on the role and future of communities on the university level. Their findings are further discussed by Goodchild (2014).

In the section *Teachers' and academics' research*, examples of research conducted in communities of practice are presented.

Influence of cooperation on participants in communities of practice

In (Wagner, 1997), co-learning agreement means that researchers and practitioners are both engaged in action and reflection. When they work together, they learn something about the world of the others but also about their own worlds.

Jaworski (2005) presents a project designed to develop communities between teachers and didacticians. The communities aimed at improving the learning of mathematics in classrooms, and at studying the processes, practices, issues and outcomes in and of the project. The roles and relationships of teachers and didacticians emerge as key concepts in the developmental process.

Bjuland and Jaworski (2009) focus on the perception of activities in a community from the perspective of teachers’ opinions on them.

In the section *Teachers’ and academics’ research*, two examples of the impact of cooperation in a community of practice on both teachers and academics are presented.

Teachers’ and academics’ research

Teachers researching their own practice

Let us illustrate differences in a teacher’s and a researcher’s roles on an example of a Czech teacher-researcher Hanka. Hanka is a full-time teacher of mathematics at a lower secondary school, pupils aged 12-15) with 2 years of teaching experience. She is a part-time student at the Faculty of Education of Charles University (Master level). She represents both – a teacher (this role will be referred to as *Hanka-teacher*) and a researcher (*Hanka-researcher*) in one person. The following episode from her professional life is intended to illustrate the differences in her two roles. For further details about the background experiment see (Reslová, 2019).

Hanka-teacher decided to implement CLIL[1] in her lessons of mathematics. She had limited experience with using CLIL in teaching from the CLIL course at the Faculty during her pre-service teacher education. She had doubts about suitability of long-term use of CLIL in mathematics.

Hanka-researcher formulated her research question: Is the use of CLIL in school mathematics beneficial for both, pupils and teachers? She decided to research her own teaching (action research). She chose the topic ratio and direct and indirect proportionality with her 13-14-year old pupils. She searched for suitable ideas in scientific publications from mathematics education and the area of CLIL.

Hanka-teacher decided about the length of the experiment (2 months) and the number of lessons conducted as CLIL lessons (1 out of 4 per week). She designed lesson plans (practicing the concepts and problems from the topic).

Hanka-researcher conducted a deep a priori analysis of the problems planned to be used in her research, including possible pitfalls. For investigating the research question she decided to use the mixed research design.

Hanka-researcher and *Hanka-teacher* (Hanka in both roles at the same time) selected the experimental and the control classes. She made a summary of what *Hanka-teacher* knows about her pupils. She prepared tests used at the beginning and end of the research episode.

Hanka-teacher conducted the 4-week teaching experiment in the two parallel classes (one with and one without CLIL lessons). In agreement with *Hanka-researcher*, she collected the following data:

[1] CLIL (Content and Language Integrated Learning) refers to the teaching of a non-linguistic subject such as mathematics through a foreign language. CLIL suggests an equilibrium between content and language learning. Both are developed simultaneously and gradually, depending on the age of pupils and other variables.

her lessons plans and all prepared materials, her field notes and pupils' tests from both experimental and control classes.

Hanka-researcher analysed the field notes. She prepared and evaluated the questionnaire survey among pupils, focusing on their opinions on CLIL lessons. Then she evaluated the experiment and answered the research question.

Hanka-researcher and Hanka-teacher compared her expectations with classroom reality. She proposed modifications for future use of CLIL. She formulated the following requirements for successful use of CLIL:

- detailed lesson planning,
- sufficient language scaffolding,
- use of topics and activities that are motivating for pupils,
- starting lessons with simpler activities,
- use of discussions in the foreign language as a motivating element.

She formulated the answer to the research question: Use of CLIL in school mathematics is beneficial for both, pupils and teachers.

A similar example is published in (Novotná, Lebethe, Rosen, & Zack, 2003).

The presented example opens new questions, e.g.

- Should all teachers conduct teacher research?
- Should faculties of education prepare teachers to conduct education research?

Cooperation of teachers and academics – example of good practice of permanent cooperation between all teachers form the school and academics: COREM (1973-1999)[2]

COREM, Le Centre d'observation et de recherche sur l'enseignement des mathématiques (school Jules Michelet, Talence, France) was based on Brousseau's ideas (Brousseau, 1997) with the following objectives (Salin & Greslard-Nédélec, 1999):

- To achieve the research necessary for advancement of knowledge of the mathematics education phenomena.
- To conceive and study new educational situations enabling a better acquirement of mathematics by pupils.
- To develop in this way a corpus of knowledge necessary for teacher training.

In COREM, a close collaboration of researchers from university, teacher trainers, elementary school teachers (pupils aged 3-11), school psychologists and students of didactics of mathematics took place. For finding and explaining phenomena of didactics referring to teaching and for research, two

[2] A more detailed version of the description of COREM can be found in (Novotná et al., 2003).

resources of data were collected: long-life collection of qualitative and quantitative information about teaching of mathematics at the elementary level and observations.

In Michelet, the teaching staff were ordinary teachers without any special training. Their task was to teach, not to do research. They worked in teams, three teachers for two classes. One third of their working hours were devoted to COREM. This time consisted of four types of activities: coordinating and preparing together ordinary work for pupils and discussing all problems of the school (educational, administrative, social and so on), observing work in the classroom, for research or just for ordinary feedback, participating with the researchers at the conception of the session to be observed and collecting data about pupils' behaviour in mathematics. In-service education was conducted in the form of a weekly seminar on the topics suggested by the teachers.

Daily mathematics activities were designed in collaboration with one teacher trainer from IUFM (Institut Universitaire pour la Formation des Maîtres) who monitored mathematics education in the school during the whole school year, was the guide for mathematics content and guaranteed that the research would not impair usual educational activities of the school. The interactions of researchers with the observed class were institutionally adjusted. The teacher had the final say about what would be done, if the team did not succeed to find a consensus. The detailed analyses of the teaching units were conducted by the whole team, including the teachers.

Observations were of two types: observations of sequences of lessons prepared in cooperation with a researcher and observation of lessons prepared by the team of teachers on their own.

- In the first case, the researcher was responsible for developing sequences of lessons and presented them to the teachers. If the project was accepted by the team, the team elaborated teaching sequences. It was the teacher who was exclusively responsible for what happened in the classroom. He or she had the right to make decisions different from the original plan. After the observed sequence, its immediate first analysis was conducted. In this analysis, all participants reconstructed all the events of the session in the following order as precisely as possible: First, the teacher summarized what was or was not good and why from their point of view. Then the team discussed issues explainable by the conditions and looked for candidates for phenomena for whom research was developed. The discussions provided the researcher with a considerable amount of additional information.
- In the second case, regular weekly observation of a series of lessons that had not been prepared with a researcher served to find and explain the contingent decisions (both good and bad) of "all" teachers.

In COREM, teachers and researchers were members of one team at least in the preparatory phase. Their roles were different. But the teacher always made the last decision. The successful functioning of COREM depended on collaboration of all participating persons as well as much administrative and managerial work. COREM greatly contributed to the development of the Theory of Didactical Situations in Mathematics (Brousseau, 1997).

Cooperation of teachers and academics – two examples of the impact of cooperation in a community of practice on both teachers and academics

A) *Improvement of pupils' culture of problem solving – teachers' change*

The research presented here is based on a longitudinal experiment focusing on how to understand, study and improve pupils' school culture in case of problem solving in mathematics education (Novotná, 2009). The situations were analysed within the frame of the Theory of didactical situations (Brousseau, 1997). The aim of the experiment was to study the tools that can transform pupils' school problem solving culture and to evaluate didactical effects in the perspective of a mathematical activity. It required a close cooperation of academics and teachers. In the research, the impact of activities on all three participating groups (teachers, academics and pupils) was followed.

In this text, the focus is on the results related to the changes in the approaches of participating teachers and academics. For further details about the background experiment see also (Novotná, Brousseau, Bureš, & Nováková, 2013.)

The experiments were conducted several times, during three school years, in Czech and Slovak schools with 12-14-year old pupils. The implementation as well as the analysis and evaluation of the experiment were done in cooperation with secondary school teachers from Prague, the Czech Republic. At least three teachers participated in the experiments every year.

The research was organised in five stages. The aim of the 1^{st} stage was to propose a different view on mathematical problems to the pupils: from a problem seen as a tool for their evaluation towards a problem seen as something to be evaluated by the pupils: the evaluation criteria were difficulty (a subjective criterion), length of text (an objective criterion), length of solution (a criterion related to the used solving strategy), attractiveness, comprehensibility and usefulness (three subjective criteria). From teachers, this stage required change of their approaches to teaching because there is no correct or incorrect answer.

The aim of the 2^{nd} to 4^{th} stages was to draw pupils' attention to a mathematical model without teaching what it was, allow them to discover problems of the same type, and to realize how understanding of these problems and their similarity could help them solve other problems.

The aim of the 5^{th} stage was to summarize and precise the knowledge the pupils acquired in the 2^{nd} to 4^{th} stages.

The teachers' responsibilities were selecting problems for their classes to be solved during the experimental stages, collecting materials produced in their classes, mediating discussions during the episodes and active participation in team discussions. The whole experiment required a minimal participation of teachers in pupils' discussions.

The findings from the research show that success in changing pupils' relationship to solving problems requires not only deep teachers' involvement in realisation of the designed activities but also their active involvement in the project design. This change of their role goes hand in hand with the change of their pedagogical approaches and beliefs. Changes in all aspects were detected in:

- the teachers' ability to design and organize efficient a-didactical situations[3] in their classes,
- their ability to analyse situations, evaluate their course and results and distinguish between the rules of the situation and contingency,
- their active involvement in designing, implementation and analysis of the research in collaboration with researchers,
- their ability to function successfully in two different roles, a teacher and a researcher.

The findings are based on teachers' self-reflections and researchers' observations. A significant increase in the teachers' autonomy was observed. During the preparation and implementation of the changes in the experimental settings, the teachers gradually took the roles of persons who actively influence the stage design and used problems. This was the outcome not only by their experience from conducting the experiment stages but largely also by their participation in the team meetings, where the experiences and preparation of the following steps were discussed.

B) *Impact of teachers' participation in research on academics*

The presented findings are a part of a three-year research project GAČR P407/12/1939 *Development of culture of problem solving in mathematics in Czech schools*. The goal of the research project was development of a theory of mathematics problem solving with focus on the role that heuristic strategies play in development of pupils' culture of solving problems. Pupils were led systematically to use a suitable heuristic strategy when they come across a problem they cannot solve using "school solving algorithm" (Eisenmann, Novotná, & Přibyl, 2014; Novotná, Eisenmann, & Přibyl, 2015). For further details about the background experiment see also (Eisenmann, Novotná, Přibyl, & Břehovský, 2015).

In short-term (3 month) and long-term (18 month) experiments, lower and upper secondary pupils were introduced by their teachers to heuristic strategies that they rarely or never came across in usual lesson but were very effective and useful in problem solving. Among other, the impact of the work of the team of teachers and academics on both of the participating groups was studied. The following changes are reported, based on interviews and observation data from the collaborative work with the teachers over the period of the whole experiment and on the basis of analysis of the structured interviews. The teachers (Novotná, Eisenmann, & Přibyl, 2015)

- lowered their demands on accuracy and correctness in their pupils' communication and recording of solving procedures in favour of understanding the problem solving procedures,
- showed more tolerance to a variety in pupils' solutions,

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[3] A-didactical situation is a situation whose objective is to enable pupils to develop their knowledge independently, without an explicit teacher's intervention (Brousseau, 1997).

- acknowledged a change in their teaching towards constructivist and inquiry-based approaches,
- grew more interested in pupils’ solving processes while solving problems.
- one of them reported that she started to think how to eliminate the pervasive sense of failure (e.g. she decided to use group work more often).

One of the most important results in this area was that most of the participating teachers started to pose their own problems with the aim of making their pupils understand the various strategies better.

When analysing the impact on academics, the following was discovered. Cooperation with teachers helped researchers significantly to:

- precise the experimental settings,
- analyse the project results.

When analysing the research team, it turned out that the division of roles and responsibilities moved from a significantly unbalanced one with most responsibility in the hands of the academics towards real cooperation with a clear division of responsibilities where everybody brought their expertise into play.

Concluding remarks

In the text, only a small part of important questions concerning the roles and cooperation of teachers and academics was discussed. We deliberately did not try to contribute to the long-lasting discussions about the position of teachers’ research in the domain of educational research. We also did not analyse the following questions: Are there differences in the research results if the direction of research design is teacher $\Rightarrow$ researcher or researcher $\Rightarrow$ teacher? If so, what are the main differences?

Let us finish with a question closely linked to the previous text: What are the benefits of close cooperation between teachers and academics? This is a broad question which can be divided into three sub-questions:

Do teachers need the direct presence of an academic during their teaching? The answer is NO. Here are the reasons: Good teachers do their important work excellently without such a close collaboration. The answers to theoretical research questions do not have a direct impact on daily work of a teacher; a teacher cannot use them in everyday situations in the classroom, in a specific situation that happens.

What are the possible benefits for the teacher of a teacher and academic in direct cooperation? Teachers can find the answers to questions that they face in their everyday teaching in academics’ results and then implement them into their teaching. But in real situations, a teacher’s reactions are a response to a specific situation where the immediate decision may be influenced by theoretical results but it is always fully “in the hands” of the teacher. A “blind” application of research results in teaching is dangerous and must be avoided.

Do academics in education need the direct cooperation with one or more teachers? The answer is YES. Academics need teachers for finding answers to research questions. They need direct contact with teachers and genuine access to the reality of teaching.

As our experiments show we can conclude that cooperation of teachers and academics is beneficial for both groups as well as for pupils.

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Using a variety of methods for mathematics education research

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In this paper, I will discuss different methods I have used in my research into mathematics education. My particular research interests have included children's understanding of mathematics and the role of visual representations, assessment in mathematics education and professional development of mathematics teachers. Reflecting these interests, I will discuss three research methods that I have found particularly useful in these areas of interest. Firstly, I will discuss research that I carried out in England using eye-tracking methods to examine primary children's understanding of visual representations of multiplication (Bolden, Barmby, Raine & Gardner, 2015). I will mainly focus on the use of eye-tracking as a qualitative video tool to find out how children examine mathematical representations and what we can infer about their understanding from the resulting videos. Secondly, I will discuss research work using a comparative judgement to assessing pupils' understanding in mathematics. Comparative judgement can be used to assess more 'nebulous' constructs in mathematics education research such as 'understanding'. I will discuss what comparative judgement is and how I use it to assess children's understanding, and also what qualitative results can be obtained in terms of progression in children's understanding in particular areas of mathematics education.Finally, I will discuss use of Design-Based Research (DBR) methodology to develop a professional development intervention for secondary mathematics teachers in an informal settlement in Johannesburg, South Africa. I will particularly emphasize how DBR develops practical interventions and theoretical perspectives which take into account and apply to particular contexts – in fact the recognition of context being an important part of this research.

Keywords: research methods, eye-tracking, comparative judgement, design-based research methodology.

1. Introduction

In this presentation, I am going to talk about three different research methods that I have used in my work in mathematics education research. I have been researching mathematics education for about 15 years, and what I would like to convey in this talk is what I have learnt about using a variety of research methods in my work. I present these methods not because I think they are particularly better than other methods that one could use; rather I highlight these simply because they have been interesting to employ in my research and that they illustrate the advantages of using different approaches.

I would like to firstly provide some context for my 'journey' through different research methods. My undergraduate degree and my PhD were in physics, and I began my career in education research from quite a quantitative standpoint. I then became involved in teacher training for teachers of primary mathematics and I became interested in how young children make sense of and understand mathematics. I spent many years at Durham University in the UK, but then I was very fortunate to work for three years at the University of The Witwatersrand in Johannesburg in South Africa, researching mathematics education in quite a different context for me. Following on from my experiences in South Africa, I now work for an assessment company called No More Marking who specialize in an assessment approach called comparative judgement.

In the different stages of my research career, I have been involved in research employing different methods appropriate to the studies we were carrying out at the time. Therefore, the three research methods that I am going to briefly describe in this talk are:

- Eye-tracking research;
- Comparative judgement; and
- Design-based research.

In describing each approach, I want to try and summarize:

- What we did in the research;
- A description of the method;
- What we found out in our studies; and
- Why I feel excited about the research approach.

Of course, the talk will provide just a very surface overview of these approaches. But in talking about these and why each one was useful for the particular study, I hope to be able to convey to you in the conclusion of the paper why using a variety of methods is useful for our work on mathematics education research.

2. Eye-tracking research

2.1 What we did in the research

The first approach I would like to talk about is the work we carried out using eye-tracking (Bolden, Barmby, Raine & Gardner, 2015). This research was part of our work on how young children make sense of and understand mathematics. The broad aim of the study was to investigate how young children view and interpret different mathematical representations. More specifically, we used an eye-tracking system to record children's visual attention with a variety of representations of multiplication. Children were asked to look at PowerPoint slides showing a symbolic and a visual representation side-by-side and simply asked if the calculation matched the picture (Figure 1). Three different types of visual representations were used: a ‘groups of’ representation (groups of strawberries); an array representation; and a number line representation. There were 18 slides in all (six of each visual representation) with some matching symbolic and visual representations, and some non-matching slides. Nine Year 5 children (four boys, five girls, aged 9-10 years of age) from a primary (elementary) school in the North East of England were asked to complete the test during school time. The children represented a range of abilities across the mathematical domain (three of higher ability, three of middle ability and three of lower ability) and were selected accordingly by their class teacher.

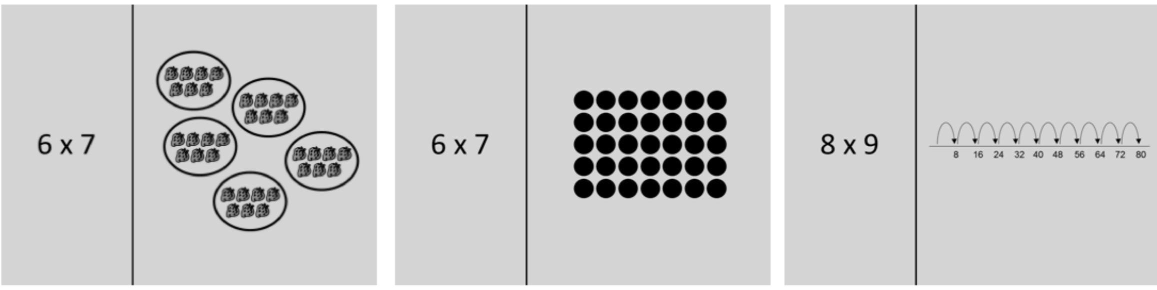


Figure 1: Children were asked whether the 'picture' matched the calculation in each case

2.2 A description of the method

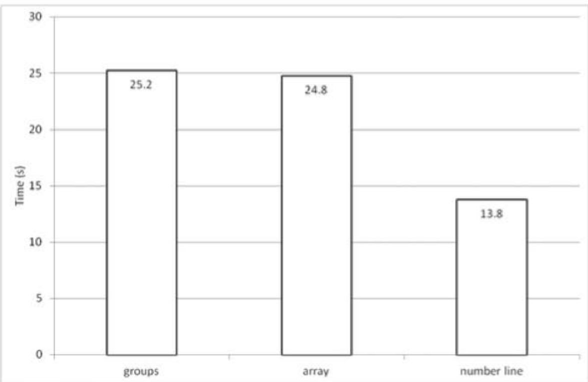
A table-mounted video-based corneal reflection eye-tracker was used (Duchowski, 2007). This system had the advantage of being portable (the eye-tracking ‘bar’ positioned below a monitor, both connected up to a laptop system), and so the data collection was carried out in a spare classroom in the school rather than in a laboratory setting (Figure 2).

Figure 2: Eye-tracker set up in the classroom

Mangold Vision eye-tracker software was used on the laptop to display the stimuli and to record the eye-tracking data collected through the infra-red tracking of where the pupils were looking on the monitor screen. The set up collected both quantitative and qualitative data for our study. Quantitative data that was captured by the eye-tracker and recorded by the software was the time spent by the pupil on each slide and also the proportion of time on each area of interest specified within each slide (for example time on the symbolic calculation or the time on the visual representation). In addition, we were also able to carry out a qualitative analysis of the video recordings of each child's gaze trajectory during each representation. By looking at the movement of the pupil's gaze around the representation, we were able to make inferences on how the pupil was interpreting a visual representation, and so allowing a categorization of the different approaches to interpretation adopted by the children.

2.3 What we found out in our studies

A full description of our findings is of course provided in our paper. Briefly though, what we found out from the quantitative data was that children spent the smallest amount of time on the number line representation slides, but were also least likely to correctly say whether the symbolic and visual representations were matching



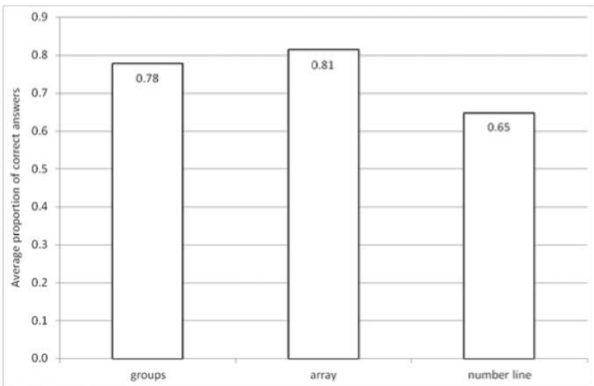


Figure 3: Mean time per representation (left) and proportion of correct answers (right)

More interesting perhaps was the qualitative data. We saw from the videos of the pupils' gaze trajectories that one reason for incorrectly saying whether the number line representation matched was that pupils quickly went to the final number on the number line. For example, a symbolic representation of 8×3 and a number line representation of 6×4 might be judged to be matching by the pupils. For the other representations as well, the video recordings showed whether a pupil had an understanding of the multiplicative structure within the visual representation. For the groups of representation, some pupils counted one group and then the number of groups, other pupils counted every element of all the groups. In the array, some pupils simply counted the numbers of rows and columns, others counted all the elements. We were therefore able to conclude whether a pupil had the multiplicative understanding associated with a visual representation.

2.4 Why I feel excited about the research approach

Of course, the eye-tracking approach provided plenty of quantitative information with which to carry out our analysis. More exciting for me though was the qualitative data. In trying to gain access to children's mathematical thinking during tasks, the most common approach has been the use of task-based interviews. However, Davis (1984) highlighted the difficulties with task-based interviews that interviewers might unconsciously encourage pupils to succeed in the task given. In addition, although task-based interviews do allow us to probe deeply into children's understanding, this probing may result in the children reflecting the researcher's ideas rather than their own (see for example Schindler and Lilienthal, 2018). The use of eye-tracking technology therefore allows us to largely avoid these difficulties, providing an alternative approach to examining mathematical thinking.

3. Comparative judgement

3.1 What we did in the research

The next research approach that I would like to talk about, and which I am involved with in my current position, is the use of comparative judgement in mathematics education research. Comparative judgement is an analytical process which can be used to assess more open-ended responses:

> *"uses professional judgement by teachers to replace the marking of tests. A judge is asked to compare the work of two students and simply to decide which of them is the better. From many such comparisons a measurement scale is created showing the relative quality of each student's work; the scale can then be referenced in familiar ways to generate test results."* (Pollitt, 2012, p. 281)

The process is based on the law of comparative judgement put forward by Thurstone (1927) for qualitative comparative judgments between stimuli. This approach comes in useful in mathematics education where rather than looking at simple right or wrong answers, you want to assess more open constructs such as pupils' understanding in mathematics or pupils' problem-solving approaches in mathematics. Dr Ian Jones and colleagues at Loughborough University in the UK have been a big proponent of this approach in mathematics education in recent years. Comparative judgement offers an alternative to traditional educational testing based on scoring rubrics (Bisson, Gilmore, Inglis & Jones, 2015).

3.2 A description of the method

For the study described here, the No More Marking comparative judgement website[4] was used to collect and process the data (Figure 4).

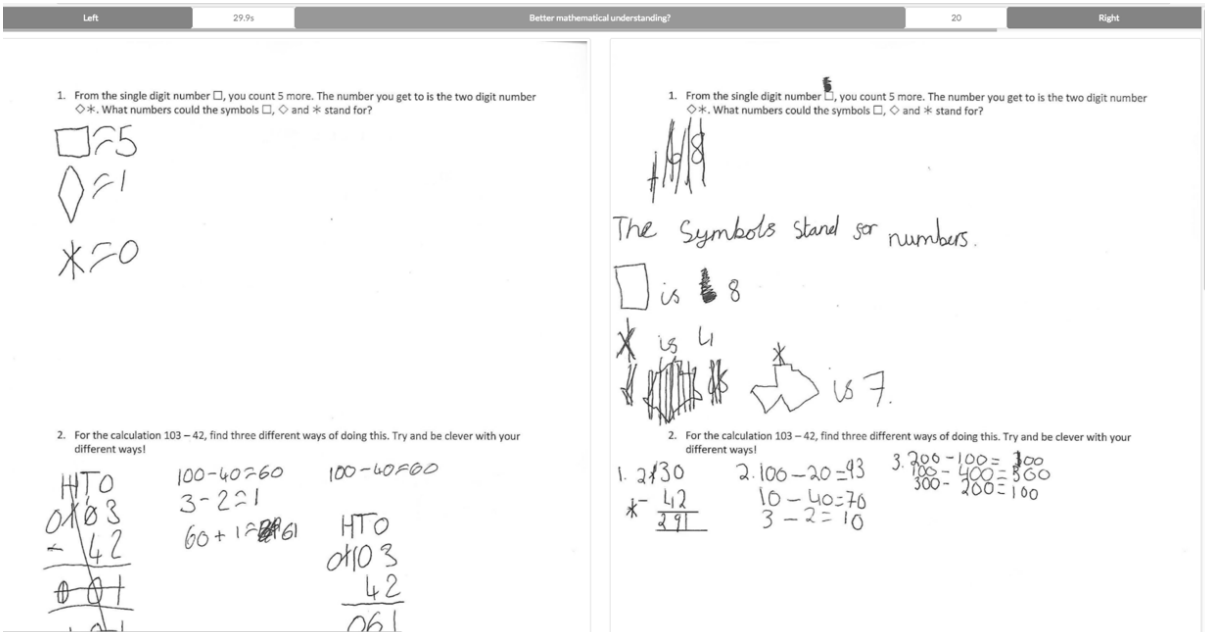


Figure 4: Example of a judgement screen from a maths CJ assessment

Pupils' responses to mathematics questions were presented side-by-side on the screen. Using this system, we carried out a primary mathematics trial for Year 5 (age 9-10) pupils. The trial involved 62 schools and 3247 pupils. The work of each pupil was judged by teachers across the schools taking part. Each pupil had been asked to answer four questions on each assessment sheet (Figure 5) and the judges were asked to choose in each case the response showing the better mathematical understanding. Using the comparative judgement approach, all the pupils were placed on a scale of 0 to 100. The overall reliability of the assessment was 0.81.

[4] www.nomoremarking.com

3.3 What we found out in our studies

1. From the single digit number A, you count 5 more. The number you get to is the two digit number BC. What numbers could the letters A, B and C stand for?
2. For the calculation 103 - 42, find three different ways of doing this. Try and be clever with your different ways!
3. A new pupil next to you doesn't know what multiplication is. How would you explain multiplication to them? You can use pictures if you wish.
4. You have four numbers: 3, 4, 5 and 6. Using two of the numbers, explain how you would make the smallest fraction possible (for example you could make 3/5).

Figure 5: Questions from the Year 5 primary maths assessment

Using traditional criteria-based approaches to assessment, it would have been very difficult to allocate scores to the pupils' responses to the above questions if we were to consider more than the correct/incorrect nature of the responses. Using comparative judgement however, all the pupils were allocated a score on the scale. In addition to the quantitative scoring of the pupils though, a qualitative analysis of the answers was carried out as well. Having ordered all the scripts using the comparative judgement process, 10 scripts were selected from the range of answers of the cohort of pupils, selected at intervals of 10 on their score (on the scale of 0 to 100). Having selected these 10 scripts, we analyzed each question to show the progression in mathematical understanding across the scripts. Properties of each answer were identified and combined into overall characteristics which we could use to analyze the different answers. By looking across these characteristics and how they progress one to another, this provided diagnostic information on how we could move pupils on from where they are. As an example, let us look at the responses for the fraction question (question 4) in Figure 5. The resulting table showing the properties of the answers for each of the 10 scripts.

Script number	No coherent answer	Incorrect answer	Partial explanation	Correct answer	Explanation given	Equivalent fractions	Alternative representations	Percentages used
#1	✓							
#2	✓							
#3		✓						
#4		✓						
#5		✓	✓					
#6				✓	✓			
#7		✓				✓		
#8				✓			✓	
#9				✓			✓	
#10				✓				✓

Table 1: Analysis of the fractions question

We can see that the properties of the answers start at the lower scoring scripts with no coherent answers. We then move to correct answers but then we can move further with explanations being provided, and further with alternative representations of fractions. As an example, Figure 6 showsanswer towards the upper end of the scale generated by the comparative judgement process.

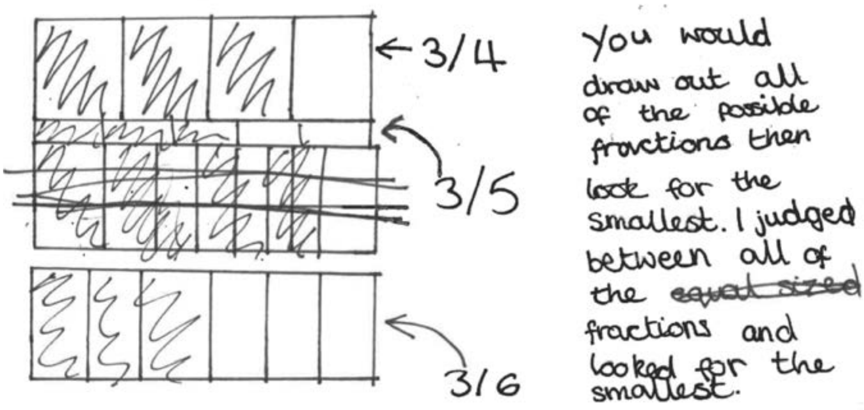


Figure 6: Example answer at the upper end of the scale

3.4 Why I feel excited about the research approach

From the perspective of assessing understanding in mathematics, I feel that the analysis shows the potential of this assessment approach, the results of which are not inconsistent to research's views of understanding (for example, the quality of explanations and reasoning, and the drawing upon of multiple representations). Comparative judgement can therefore provide an approach to assess and research what we might think of as more 'nebulous' constructs such as 'understanding' or 'problem solving':

> *"One potential contribution that CJ might offer education is its suitability for assessing nebulous constructs that are deemed important but which are difficult to specify comprehensively in mark schemes"* (Jones & Inglis, 2015)

From a practical perspective as well, the ordering by comparative judgement provides teachers with an indication of next steps in developing students' understanding. This progression, because of the comparative nature of the approach, is not necessarily limited by ceiling effects that might exist with criteria-based approaches, and teachers can exemplify to pupils how they are able to extend their understanding of a mathematical topic beyond that of simply providing a correct answer.

4. Design-based research

4.1 What we did in the research

The final approach that I would like to briefly talk about is design-based research (DBR). For this, I am very much drawing upon the work of my former PhD student, Brantina Chirinda, who used DBR in the context of developing professional development interventions for teachers in a particular context in South Africa.

Wang and Hannafin (2005) describe the DBR approach in the following way:

> *"a systematic but flexible methodology aimed to improve educational practices through iterative analysis, design, development, and implementation, based on collaboration among researchers and practitioners in real-world settings, and leading to contextually-sensitive design principles and theories"* (p. 2)

DBR is characterized by this iterative approach of developing educational practice, following these kinds of steps as part of the iterative cycle:

- Analysis of the problem;
- Design and development of solution for the problem;
- Implementation of solution;
- Evaluation of the implementation;
- Leading to further reflective analysis of the problem.

The acknowledgment of context and the adaptation of both practice and theory in light of what works in the context is a vital aspect of DBR. To do this, researchers work as a team with practitioners to provide solutions that practitioners face in a particular educational context. Therefore, DBR should generate valuable educational interventions and useful theory that is applicable to given contexts.

4.2 A description of the method

I will describe the work carried out in the particular study of Chirinda and Barmby (2017). The main objective of the study was to design an effective professional development intervention to support mathematics teachers in the teaching of mathematical problem solving. The context of the research was schools with large classes of pupils in an informal settlement outside Johannesburg. The study began with a baseline investigation with 31 teachers at 20 schools in the district of interest. The teachers were asked to complete an open response questionnaire on their views of teaching problem solving in order to find out what might be the issues for them in their mathematics lesson in that context.

Following on from the baseline investigation, two teachers from one school took part in three workshops over a six-month period focusing on incorporating Polya's problem solving steps in teaching. The data collection looked at both the process of the intervention and how it was working for the teachers involved, and also the impact of the intervention on teachers and pupils.

Data collected (process):

Classroom observations

Interviews with teachers

Data collected (impact):

Math assessment results for pupils (pre and post)

Problem solving inventory/questionnaire for pupils (pre and post)

Task-based interviews with pupils (second cycle only)

Following the first iteration of the intervention and the analysis of the data collected from this, the iteration was repeated with another two teachers in a second cycle.

4.3 What we found out in our studies

In terms of the impact of the intervention developed in the study, for pupils, it was found that over the six-month period of the intervention, they made statistically significant gains in the mathematics assessments, and scored more highly in the problem solving inventory (i.e. more likely to carry out actions in line with Polya's steps during their tackling of problems). For the teachers as well, they identified the following benefits of the intervention:

- Greater focus on 'understanding the problem' - including tackling the issue of a multi-lingual classroom;
- Greater focus on collaborative learning;
- Greater focus on reflection on approaches by pupils.

Perhaps more interestingly, what also came out strongly from the DBR approach was the overriding issue of the importance of context and responding to that context. For example, the design principles developed for the intervention included the following:

- The importance of a respectful relationship between the researcher and teachers;
- Taking into account what teachers see as a mathematical problem;
- Incorporating support for teachers on how to implement mathematical problem-solving pedagogy in a multilingual context.

These principles applied particularly to this context – for example, incorporating language issues into the teaching of problem solving was vital in classrooms with several first languages being spoken.

4.4 Why I feel excited about the research approach

In my brief experience of being involved with DBR, one of the exciting aspects of the approach for me is that it provides an alternative to the clinical randomized controlled trial approach to research that is very often applied to educational interventions. That is not to say that RCTs are not important in educational research; they mostly certainly have their place in assessing the effectiveness of well-developed interventions. However, for less developed interventions, or for interventions being applied to new contexts, I feel that DBR is a much more useful tool for the development of practice and knowledge in education. The reason for this is that, as has been emphasized above, context is central to the development of interventions in DBR and possibly theory. DBR provides an iterative approach to developing useful educational practice and knowledge in those particular contexts. The results of the process are not simply the measured impact of the intervention, but also information on the process of the intervention and how this can be fed into the design principles for the intervention in the next iteration. In order to look at both impact and process, the other exciting aspect for me about DBR, and relevant to this current talk, is that this 'broader' view of the research process requires a greater range of research methods. DBR therefore also encourages the researcher to broaden their research perspective, to perhaps mix the quantitative impact data collection with the more qualitative process data collection.

5. Concluding thoughts

The aim of this talk was to simply talk about a variety of research methods that I have used in my work, highlight their different uses, and perhaps to encourage others to consider a wider variety of research approaches in their work, whatever these research approaches happen to be. With regards to this latter point, if there are themes to be drawn out from the three different methods described in this paper, the first theme would be that using a broader range of methods has provided me with possible ways of studying ‘hard-to-reach’ constructs such as ‘understanding’ and ‘thinking’. This is particularly true for the eye tracking and the comparative judgement approach to assessment. The second theme would be the advantages of mixing quantitative and qualitative approaches to data collection to provide greater insights in mathematics education research. To emphasize the importance of this point further, we can look at some research on the prevalence of mixed methods research (i.e. mixing both qualitative and quantitative approaches to data collection during a research process) in mathematics education research. Hart, Smith, Swars and Smith (2009) looked at 710 research articles in mathematics education published in six educational journals during 1995-2005. Their finding were that:

- 50% used qualitative methods
- 21% used quantitative methods
- 29% used mixed methods

We can therefore see that only around a third of research studies in mathematics education utilize the possible advantages of mixing research methods. Taking this point further, and looking at the types of mixed methods used, mixed methods can be separated into different types which include:

- concurrent designs (quantitative and qualitative at same time);
- confirmatory sequential design (quantitative leading to qualitative);
- exploratory sequential design (qualitative leading to quantitative).

A concurrent design might be, for example, like the eye tracking design described above. A confirmatory sequential design might be carrying out say a quantitative attitude study using a questionnaire and following up with interviews to try and explain the results. An exploratory sequential design might be to carry out interviews first to collect qualitative data that supports the construction of a quantitative-approach questionnaire. Now although this was not very rigorous, I carried out my own informal study into the types of mixed methods approaches used in recent years in mathematics education research. I found that almost all these mixed method studies in mathematics education used a confirmatory sequential design – using qualitative data to explain further the findings from quantitative data in a study. What I am trying to illustrate here, therefore, is that as a community, we perhaps do not utilize the range of approaches that we could do to in mathematics education research. And I think this is a pity. I hope that I have showed to some extent the usefulness of mixing approaches and using a variety of methods in research. But in addition to the usefulness, there is also the enjoyment that I have gained in learning about and trying to apply these different methods which has been a big part of my journey in mathematics education research. I therefore hope that I have also conveyed this enjoyment, and that I have encouraged you in considering a greater variety of research methods in the future.

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Educational standards in mathematics at the end of secondary education - Analysis of students' achievements

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Standards for general secondary education are based on competencies that will enable students to successfully respond to various life challenges (educational, social, cultural, interpersonal, practical, etc.). Three levels of achievement are defined for each competency - basic, intermediate and advanced. Three levels are cumulative and embedded one in another so that students at the advanced level fulfill requirements from other levels. Methodology for the development of educational standards for the end of secondary education. In this paper we present the results of the empirical examination of students' achievements in Mathematics.

Keywords: Educational standards, Student achievements, General secondary education, Mathematical competences.

1. Educational standards in secondary education in Serbia - context

The education system in the Republic of Serbia consists of three interconnected basic components: curricula, teaching and learning, and evaluation of the achievement of learning outcomes as determined by educational standards. The educational standards are determined on the basis of the general outcomes of education and upbringing - which are the result of the whole process of education. The strength of an educational system is largely reflected in the existence of a developed and efficient system for monitoring and evaluating quality. In a complex system of evaluation of the quality of education, students' achievements have a particularly important place - the knowledge and skills that students acquire during a certain phase of education, represent the quality of "output", and therefore the most important measure of the achievement of the set goals of the educational process (Pešić et al., 2009).

There are some opposing views when introducing educational standards into the education system. One is that standards in education will lead to too much formalism in schools, while the other argues that schools cannot set goals and effectiveness on their own until they fully understand the standards in education (Judo C., 1917).

The educational standards for the end of secondary general education in Serbia have been set for all students in the education system, for those in general secondary education and for those in vocational education. It is far better to choose our own standards that consist of our one values and local socieconomic realities. (Ellman N., 1979)

By developing mathematical competences, it is also necessary for students to develop cross-curricular competences. Thus, the student integrates the different knowledge and skills that he / she has acquired within different subjects. Essentially, working on general and cross-curricular competencies is not competitive with work on content and competencies that are directly related to particular subjects. On the contrary, cross-curricular competences represent a step further in understanding the material and applying the lessons learned, and responsibility for their development rests with all teachers and school subjects. This means that supporting general and cross-curricular competences requires joint planning at the school level, implementation of interactive and active forms of learning, as well as greater autonomy of schools and teachers in implementing educational outcomes. (Rosic, 2015)

The starting points in defining standards, in addition to the general goals and tasks of mathematics, concerned finding real-life examples, linking mathematics to the natural and social sciences as well as its application in solving practical problems. Emphasis is placed on the functionality of the acquired knowledge so that every student is able, at the end of this cycle of education, to identify and understand the role that mathematics plays in the modern world. Every student should perform well-founded mathematical assessments and have knowledge of mathematics to help him / her become a constructive, interested and reflexive citizen.

Mathematics subject standards are based on general subject and subject specific competencies. The general subject competence, within the standard, is to enable the student to think mathematically, to acquire mathematical knowledge and concepts, to critically analyze processes and to improve them, to understand how processes lead to problem solving. The student should develop an exploratory spirit, the ability to think critically, formally and abstractly, as well as deductive and inductive thinking. Furthermore, the student must learn and understand thinking by analogy, develop the ability of mathematical communication and adopt positive attitudes towards mathematics and science in general, apply mathematical knowledge and skills to solve problems in the natural and social sciences and daily life, but also be able to use the acquired knowledge and skills in further education.

Basic level	Defines the level of achievement in certain mathematical competences that student needs to adopt in order to actively and productively participate in different areas of life (social, economic, educational, family, personal, etc.)
Intermerdiate level	Defines the level of achievement in certain mathematical competences that student needs to acquire in order to successfully continues education in various fields
Advanced level	Defines the level of achievement in certain competences that student have to possess in order to be able to successfully continues education at faculty, in area for which those competences are a particularly important requirement.

Table 1: Levels of educational standards (Rosic, 2015)

The general subject competence is divided into three levels: basic, intermediate and advanced. These levels are shown in Table 1. Specific subject competencies are classified into three domains:

Mathematical Knowledge and Reasoning, Application of Mathematical Knowledge and Skills to Problem Solving, and Mathematical Communication. All standards are divided into four major areas: 1) Algebra, 2) Geometry, 3) Sequences, Functions, Derivatives and Integrals, and 4) Combinatorics, Probability, Statistics and Financial Mathematics.

Each individual standard is identified by the code as follows **2.MA.x.y.z.** Number **2** denotes the fact that these are standards for the end of general secondary education and vocational secondary education in the part of general education subjects. **MA** is an abbreviation for the name of the subject, **x** is a level designation (1 - basic, 2 - intermediate and 3 - advanced), **y** is a designation for the area (1 - Algebra, 2 - Geometry, 3 - Sequences, functions, derivatives and integrals and 4 - Combinatorics, Probability, Statistics and Financial Mathematics), and **z** is the number of standards within a given field. For example, code **2.MA.2.1.8.** indicates that this is standard for the end of general secondary education for mathematics, at ntermediate level, in field Algebra and that it is the eighth standard in that field, at that level. Education standards can be roughly divided into sub-areas within the area, shown in Table 2.

Algebra	• Knowledge of different sets of numbers, numbering systems, operations with them, determining the values of numerical expressions and approximate values, • Transformations of algebraic expressions, proving equality and inequality, • Solving equations, inequalities, systems of equations with and without parameters, and • Logic, set operations and relations.
Geometry	• Elementary terms, properties, claims and formulas in planimetry and stereometry, • Isometric transformations, constructions and proofs, • Trigonometry, and • Analytical geometry and vectors.
Sequences, functions, derivatives and integrals	• Sequences and mathematical induction, • Functions and their graphics, • Derivations and • Integrals.
Combinatorics, Probability, Statistics and Financial Mathematics	• Combinatorics, • Probability, • Statistics and • Financial mathematics.

Table 2: Subdomains of mathematics

2. Results of testing of student achievements

In the process of creating educational standards for the subject of mathematics, they have been methodologically developed in two cycles, through the following stages:

- the stage of expert evaluation of the working groups on the content and levels of competences, according to the analysis of existing curricula, the results of previous tests;
- empirical verification phase (paper-pencil test and teacher questionnaire) - pilot and main test of student achievement;
- the synthesis phase of expert judgment with the analysis of empirical research results.

The students had math formulas available on the test paper and were allowed to use calculators.

Standard	Item code	EL	AL	%L	%M	%G	%VET4	%VET3	D	%NR
MA.1.1.1.	M301	1	2	25%	52%	21%	11%	2%	0,43	1%
MA.1.1.2.	M401	1	2	18%	52%	24%	6%	2%	0,47	20%
MA.1.1.3.	M302	1	2	47%	73%	42%	43%	10%	0,36	15%
MA.1.1.5.	M303	1	1	56%	89%	50%	36%	0%	0,42	8%
MA.1.1.5.	M601	1	2	19%	38%	14%	2%	0%	0,54	31%
MA.1.1.6.	M402	1	2	20%	37%	22%	10%	4%	0,39	21%
MA.1.1.7.	M602	1	2	27%	43%	22%	19%	7%	0,39	21%
MA.2.1.1.	M306	2	2	45%	56%	31%	23%	8%	0,29	5%
MA.2.1.2.	M101	2	2	26%	65%	28%	9%	12%	0,51	15%
MA.2.1.2.	M403	1	3	3%	26%	5%	1%	0%	0,48	51%
MA.2.1.3.	M404	2	3	6%	23%	6%	5%	0%	0,34	28%
MA.2.1.4.	M603	2	2	38%	65%	38%	20%	16%	0,47	14%
MA.2.1.5.	M102	2	2	28%	59%	11%	7%	2%	0,53	20%
MA.2.1.6.	M604	2	2	26%	49%	23%	16%	13%	0,48	26%
MA.2.1.8.	M605	2	3	7%	24%	9%	4%	7%	0,30	6%
MA.3.1.1.	M103	3	3	7%	19%	6%	10%	10%	0,33	27%
MA.3.1.1.	M304	3	2	10%	62%	26%	11%	16%	0,45	18%
MA.3.1.3.	M606	3	2	14%	39%	14%	9%	16%	0,36	42%
MA.3.1.5.	M305	3	1	49%	63%	40%	31%	22%	0,20	14%

Table 3: Student achievement in Algebra

Table 3 shows the students achievements during main testing regarding educational standards in Algebra. The table contains the following information, for each item: **Standard** - Educational standard tested by item; **Item code** - item code; **EL** - expected level of achievement; **LA** - level of achievement at testing; **%L** - percentage of high school students with social-linguistic major, who completed the item correctly; **%M** - the percentage of high school students with natural and mathematics major, who completed the item correctly; **%G** - the percentage of high school students who completed the task correctly; **%VET4** - the percentage of school students from four-year vocational secondary schools, who completed the task correctly; **%VET3** - the percentage of school students from four-year vocational secondary schools, who completed the task correctly; **D** – item discriminability expressed through correlation of the task with the overall achievement in a given subject, indicating to what extent the achievement of students who overall had high achievements in a given subject differ from those who overall had low achievements in a given subject; **%NR** - percentage of high school students who did not respond.

The items in the table are arranged in accordance to the levels of educational standards. We can notice that there are some discrepancies in the expected and obtained level of achievement of students in 11 items, out of 19. It is expected that students of high schools with major of mathematics have significantly higher achievements than other students. By analyzing individual items, we can conclude that simplest task for students was M303 (see Figure 1).

Solve the equation.

$x^2 + 12x - 13 = 0$

x_1 = ______ and x_2 = _________

Figure 1: Item M303

The students were familiar with this item from a school context and this is one of the tasks they certainly did see in math classes, so it was expected to achieve high achievement here. The students had the lowest achievement in M404 item (see Figure 2). Although the task is very simple in terms of mathematical operations, and if it is known that the students were able to use the calculator, it seems that they have not sufficiently studied the given questions.

Approximate numbers are given:

$8,71 \le a \le 8,75$ and $b = 5,04 \pm 0,01$.

Fill the table as it started.

	$\underline{p}$	$\overline{p}$
a	8,71	8,75
b		
$a + b$		
$a - b$		

Figure 2: Item M404

In the table we can see that tasks M101 and M403 belong to standard MA.2.1.2. from the middle level, but there is also a clear difference in student achievement. The reason for such a large difference in achievement within one standard may be in the very nature of educational standards, which are broad and encompass different algebra knowledge, skills and skills within one standard.

An analysis of the data, obtained for Algebra, shows that students successfully completed item M302. This item is specific, it is not common in teaching practice, so most students probably saw this type of assignment in mathematics for the first time. Bearing in mind cross-curricular correlations, students had to use the acquired knowledge and skills from physics to solve this task, and applied this in solving this problem (see Figure 3).

Barrel on the picture has dimensions *h* = 9 dm, *D* = 6,5 dm и *d* = 5 dm. The formula for volume of the barrel is:

$$V \approx \frac{\pi h}{12}\left(2D^2 + d^2\right).$$

What is the volume of this barrel ($\pi \approx 3{,}14$)?

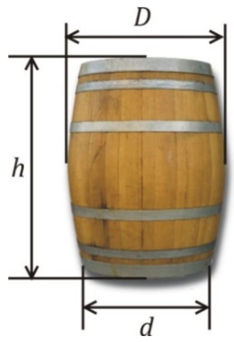

Figure 3: Item M302

Table 4 shows students achievements during main testing of educational standards in Geometry.

Standard	Item code	EL	AL	%L	%M	%G	%VET4	%VET3	D	%NR
MA.1.2.2.	M104	1	1	64%	72%	46%	28%	2%	0,39	7%
MA.1.2.2.	M201	1	3	10%	26%	8%	3%	0%	0,37	22%
MA.1.2.3.	M105	1	3	13%	32%	4%	2%	0%	0,44	38%
MA.2.2.1.	M307	2	3	0%	24%	1%	0%	0%	0,38	44%
MA.2.2.2.	M308	2	2	17%	39%	23%	13%	10%	0,24	31%
MA.2.2.3.	M309	2	2	32%	46%	23%	21%	14%	0,32	13%
MA.2.2.4.	M106	2	3	4%	24%	4%	0%	0%	0,34	33%
MA.2.2.5.	M107	2	3	6%	23%	3%	0%	0%	0,43	42%
MA.3.2.2.	M202	3	3	1%	19%	5%	1%	0%	0,35	38%

Table 4: Student achievement in Geometry

The students had best achievement in item M104, which in its complexity corresponds to the basic level (see Figure 4).

If the surface of the ball is 144π cm^2, how much is its volume?

The volume of the ball is ____________ cm^3.

Figure 4: Item M104

The standard that indicates knowledge of the surface and volume of the ball is also in the list of educational standards for the end of secondary education. However, the achievements of three-year VET students indicates that this issue is very complex for them.

Also, we can see in table that students of natural-mathematical major have had significantly lower achievements, than expected, in items at basic level M201 (see Figure 5) and M105 (see Figure 6).

In the table we can see that tasks M201 and M104 belong to standard MA.1.2.2. from the basic level, but there is also a clear difference in student achievement. The reason for such a large difference in achievement within one standard may be in the very nature of educational standards, which are broad and encompass different mathematical knowledge, skills and skills within one standard.

Draw in equilateral triangle tree line segments so that you get the correct network equilateral three-sided pyramid.

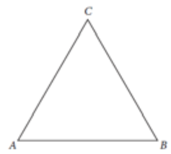

Figure 5: Item M201

The guard spotted a fire from a control tower in the National Park Kopaonik. Height of the tower is 35 m, and the depression angle (the angle under which the person watches the fire in relation to horizontal line) is 22°. How far is the fire from the foot of the tower?

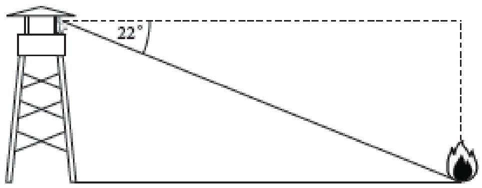

Figure 6: Item M105

These two items are different from the usual school assignments. In the first item, student are expected to estimate the surfaces of geometric bodies in 3D space, while in the second item, student are expected to estimate and calculate distances in a plane, using formulas. Having in mind overall achievements in this field, some secondary analyzes or other researches need to be done to uncover reasons andfactors for low achievement.

Table 5 shows the achievements of students in the area Sequences, functions, derivatives and integrals.

Standard	Item code	EL	AL	%L	%M	%G	%VET4	%VET3	D	%NR
MA.1.3.2.	M501	1	2	9%	45%	15%	3%	0%	0,54	28%
MA.2.3.1.	M405	2	2	32%	56%	33%	26%	22%	0,42	7%
MA.2.3.2.	M502	2	3	5%	26%	0%	1%	0%	0,54	62%
MA.2.3.3.	M503	2	2	24%	49%	40%	15%	10%	0,35	23%
MA.3.3.1.	M406	3	2	14%	44%	21%	7%	4%	0,43	21%
MA.3.3.4.	M407	3	3	4%	16%	0%	1%	0%	0,52	74%
MA.3.4.2.	M505	3	3	0%	4%	1%	1%	0%	0,28	15%

Table 5: Student achievement in Sequences, functions, derivatives and integrals

In the area of Sequences, Functions, Derivatives and Integrals, in 4 of 7 examined standards, there was a coincidence between expected and evaluated level, while for the three standards there were certain disagreements. Standard MA.2.3.1. which was tested by item M405 (see Figure 7) was best solved by the students of natural and mathematical major.

When solving this type of item, students choose various strategies for solving it. One strategy for solve this problem is to form an arithmetic sequence and to calculate largest possible sum of n numbers that is less or equal to 52. Another strategy, which we can call "brutal force", which is alsoused by students, is to count the cards in each row and then adding row by row, until the correct answer is obtained. An interesting fact is that only 7% of students did not even attempt to solve this item.

The picture shows a tower of cards that has three floors. Those with skillful hands can also build multi-storey towers. How many floors can tower maximaly have, using a deck containing 52 cards?

a) The tower may have a maximum of 5 floors.

b) The tower may have a maximum of 6 floors.

v) The tower may have a maximum of 7 floors.

g) The tower may have a maximum of 8 floors.

d) The tower may have a maximum of 9 floors.

Figure 7: Item M405

Standard 3.4.2., tested by item M505, has the worst achievements, ie. only 4 (out of 229) students completed this task correctly, although a large number of students tried to solve it (see Figure 8).

The figure shows the graph of the first derivative of the function $f : [-5,5] \rightarrow R$. Fill the line in following sentences.

The function grows at the interval ____________________.

The value of the function is the maximal point ________________.

The function has a maximal point in point ________________.

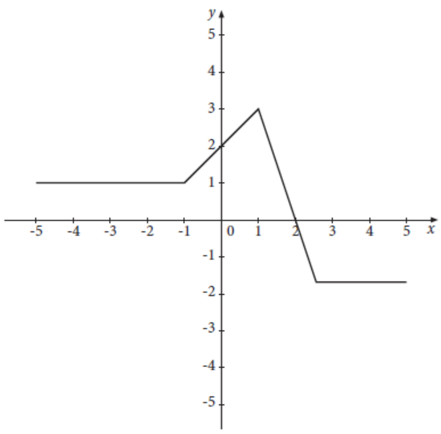

Figure 8: Item M505

Table 6 shows the achievements of students in the field of Combinatorics, Probability, Statistics and Financial Mathematics.

Standard	**Item code**	**EL**	**AL**	**%L**	**%M**	**%G**	**%VET4**	**%VET3**	**D**	**%NR**
MA.1.5.2.	M203	1	1	66%	86%	55%	48%	35%	0,40	4%
MA.2.5.1.	M204	2	2	31%	47%	28%	27%	31%	0,30	18%
MA.2.5.1.	M205	2	3	28%	35%	20%	16%	14%	0,25	20%
MA.2.5.2.	M504	2	3	6%	36%	12%	2%	0%	0,42	36%

Table 6: Student achievement in Combinatorics, Probability, Statistics and Financial Mathematics

In the field of Combinatorics, Probability, Statistics and Financial Mathematics, half of a total of 47 tested standards, have had matching between expected and diagnosed results, while other half showed different results them expected. Standard MA.1.5.2. tested by item M203 was most successfully solved (see Figure 9).

For six hours of continuosly work, machine packs 1260 boxes of some product. To get 10,000 packaged boxes, you need:

a) more than 12 hours, but less than 24 hours;

b) exactly 24 hours;

v) more than 24 hours, but less than 48 hours;

g) exactly 48 hours;

d) more than 48 hours.

Figure 9: Item M203

To complete this item, students had a lot of different ways to come up with the correct solution.

3. Conclusions and recommendations

A high-quality math program is essential for all students, as it provides them with a solid foundation and great career opportunities, in any of the areas. The examination of student achievement described here, in addition to the many good results it has produced, has shown that students have very little motivation to work when testing is not for assessment. This leads to the fact that individual expectations were well above student achievement. Having common national standards with clearly defined educational outcomes is a step towards greater school accountability but many questions remain. (Lenskaya E., 2013).

The results of this testing and the conclusions reached after the analysis largely influenced the finalization of the proposed educational standards. The data that came from this testing contained information that influenced decision making on how to improve individual standards. Testing of educational standards for the end of secondary general education was the first national testing of students in the final grades of secondary education, and since there were no benchmarks or values before, there was a significant difference in the expected and achieved achievement.

Educational standards are designed to support the basic task of mathematics education, which is to research and develop mathematics teaching at all levels, including foundations, goals and a comprehensive environment (Wittmann, 1995).

No individual or institution like to tell bad news about poor performance on tests. To the extent that this bad news stimulates instructional changes and improvements, however, it can be good new in the long run. (Anrig G., 1985). This should be considered, both when creating new or redesigning existing standards in mathematics, but also when working with students, who must meet the required standards satisfactorily.

Acknowledgment

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Reflections of future teachers of lower elementary grades on performed mathematics lessons

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In Slovenia, self-contained instruction system is applied, where elementary teacher (so called class teacher) teaches all school subjects until 5th grade, including mathematics. In order to gain some insight into class teachers' perceptions regarding teaching mathematics, we analysed reflections written by student teachers (N = 205). They reflected on their second self-performed mathematics lesson and included general pedagogy aspects (e.g. motivation, discipline…) and subject specific (in our case mathematical) pedagogy aspects (e.g. mathematical misconceptions, mathematical representations…). According to Shulmans Pedagogical Teaching Knowledge, both aspects should be included. Our results show, that approximately 40% of participants did not even mention mathematics pedagogy related aspects in their reflections. On basis of our results, we raise a question regarding necessity of additional licence for teaching mathematics at lower grades of elementary school.

Keywords: elementary mathematics specialist (EMS), general pedagogy, mathematics knowledge for teaching (MKT), departmentalized instruction system, self-contained instruction system.

Introduction

Education systems of different countries differ, among other things, also by education of teachers, who teach a specific school subject (in our case mathematics) in lower grades of elementary schools. For example, in elementary schools in China specialist often teaches mathematics, while in the USA and in most European countries the situation is very different. In these countries teachers teach all school subject, beside mathematics also mother tongue, arts, sports, natural and social science etc. Such system of teaching is called self-contained instruction system and is implemented also in Slovenia. Mathematics (and other school subjects) in the age group 5-12 years are namely taught by so-called class teachers, who during the education gain general pedagogic knowledge and also some subject specific (pedagogy) knowledge.

Shulman (1986) introduced a concept of Pedagogical Content Knowledge (PCK). PCK represents the blending of content knowledge and (general) pedagogic knowledge "into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instructions" (Schulman, 1987, p. 8). PCK is unique to teachers and differs, for example, a mathematics teacher from mathematician (scientist) (Shulman, 1987).

Several researchers upgraded and broadened PCK in different ways. Voss, Kunter and Baumert (2011) have broadened Shulmans' original definition of teachers' pedagogical knowledge by adding psychological aspects by stating that teachers knowledge "includes pedagogical and psychological aspects relating to the classroom as a social group as well as to the heterogeneity of individual students" (Voss, Kunter, & Baumert, 2011, p. 953). They introduced general pedagogic/psychologic knowledge (PPK) as "the knowledge needed to create and optimize teaching-learning situations, including declarative and procedural generic knowledge of effective teaching that is potentially applicable in a wide variety of subjects". (Voss, Kunter, & Baumert, 2011 p. 953). General PPK is general pedagogical knowledge, namely, it is not meant to be specific to a particular school subject, but to be relevant across subject (Voss & Kunter, 2013). It encompasses five components of knowledge: 1) the knowledge of classroom management; 2) the knowledge of teaching methods; 3) the knowledge of classroom assessment; 4) the knowledge of learning processes; and 5) the knowledge of individual students' characteristics (Voss, Kunter, & Baumert, 2011). The structure of general PPK is presented in Figure 1.

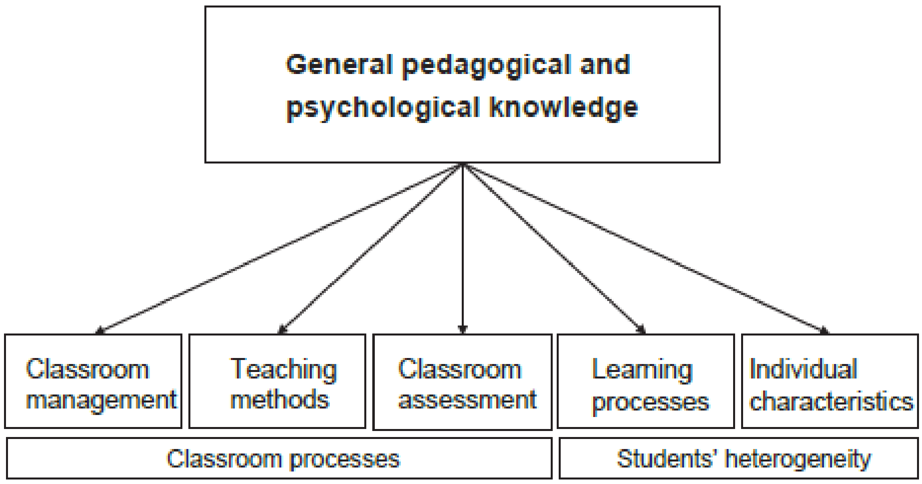

Figure 1: The conceptualization of PPK (Adapted from Voss, Baumert, Kunter, 2011, p. 954)

Teachers acquire PPK over the course of the professional career, and it is, as PCK, specific to the teacher profession (Voss, Kunter, & Baumert, 2011).

General PPK is important in all school subjects, but as Voss, Baumert and Kunter (2011) state, there is not a lot of systematic investigations of teachers' general PPK and its effect on quality of teaching and student achievement. There exist more studies, which consider specific components of general PPK, for example classroom management (Garrahy, Cothran, & Kulinna, 2005).

In the case of mathematics, Shulman's conceptualization of PCK was deepened. The concept of Mathematical Knowledge for Teaching (MKT) was introduced as mathematics specific PCK that mathematics teachers need to be able to teach mathematics. Examples of such knowledge among other things include the knowledge of explaining terms and concepts to students; providing students with examples of mathematical concepts, algorithms, or proofs; interpreting students' statements and solutions; judging textbook treatments of particular topics; using representations accurately in the classroom; etc. (Hill, Rowan & Ball 2005). MKT was found to be a characteristic predictor of the success of teaching mathematics (Hill, Rowan & Ball, 2005).

Ball, Thames and Phelps (2008) described the MKT components as follows (see Figure 2):

- Common Content Knowledge (CCK): the general mathematical knowledge, which is acquired by the majority of educated people;
- Horizon Content Knowledge (HCK): awareness of how each mathematical topic is upgrading with respect to the curriculum.
- Specialized Content Knowledge (SCK): specialized mathematical knowledge and skill, which is unique and necessary for teaching mathematics;
- Knowledge of Content and Students (KCS): the knowledge that combines knowledge of content and students,
- Knowledge of Content and Teaching (KCT): the knowledge that combines knowledge of mathematics and didactic of mathematics; and
- Knowledge of Content and Curriculum (KCC).

In Figure 2 shows a relation between the components of MKT and two Shulman's categories of knowledge: content knowledge and pedagogical content knowledge (Shulman, 1986).

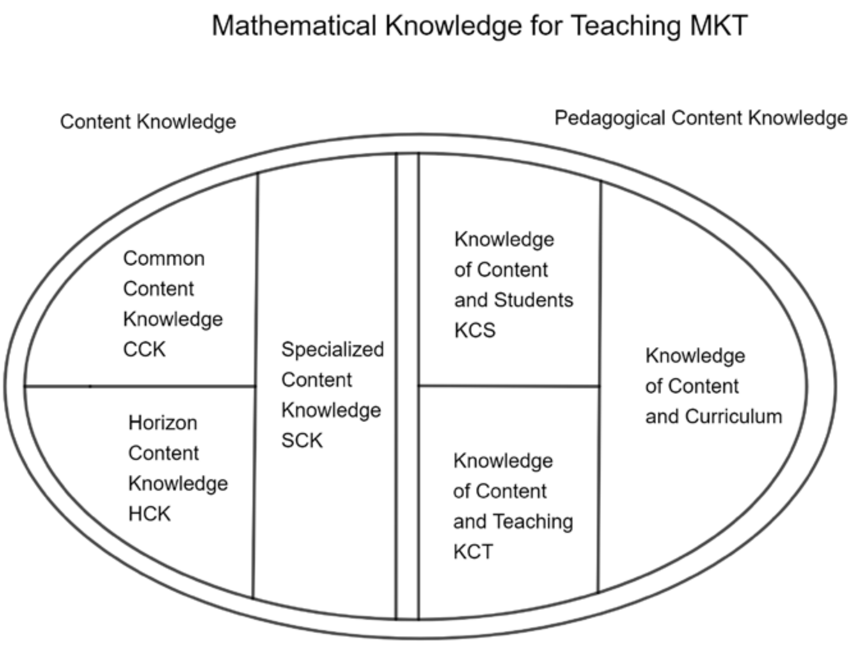

Figure 2: MKT (Adapted from Loewenberg Ball, Thames, & Phelps, 2008, pp. 403).

Subject specific knowledge and general PPK interact closely in specific teaching situations. Hence, for teachers it is not enough to master only general or only subject specific knowledge. In order to create cognitively activating learning opportunities in a subject, they need both of the mentioned components of knowledge (Voss & Kunter, 2013). Actually, already Shulman (1987) stated that content knowledge and pedagogical strategies should necessarily interact in the minds of teachers.

There arises the natural question, which component of teachers' knowledge, general pedagogic knowledge or subject specific pedagogical knowledge, is more important in the case of mathematics in lower grades of elementary school. Namely, it is not clearly known, which component should be emphasised during the education of teachers and furthermore, who should teach mathematics in elementary school: mathematics specialist or class teacher.

Interest in the question of how elementary teachers need to know mathematics to teach was thoroughly addressed by Ma (1999). In her study, Ma compared Chinese and U.S. elementary teachers' mathematical knowledge for teaching. Producing a portrait of dramatic differences between the two groups, Ma used her data to develop a notion of "profound understanding of fundamental mathematics", a concept similar to MKT in case of elementary mathematics. Maybe the reason for the differences between China and USA is also in different type of a teacher. As already mentioned, in elementary schools in China, mathematics specialist often teaches mathematics. This situation is far less common in USA and in most of European countries. In Chinese grades 1 and 2, the mathematics teacher may also teach Chinese literacy. In some places, mathematics teachers also teach science, though science is not a prominent part of the curriculum in Chinese elementary schools. However, in schools in rural or remote areas, an elementary teacher in China may teach all subjects, as is often the case in similar situations in USA (National Research Council, 2010). Parker, Rakes and Arndt (2017) explored USA elementary principals' decision-making in regards to K–5 grade level and reported that recent trends indicate a move away from self-contained classrooms and toward content-focused departmentalization in elementary schools. Lately several projects are running in USA to educate so called elementary mathematics specialist (EMS). EMS is an elementary teacher with special training and certificate for teaching mathematics. Training is designed around deepening mathematical content knowledge (for more details see McGatha, & Rigelman, 2017).).

Some research shows that significant number of elementary teachers have a lack in MKT, namely a lack of sufficient mathematical knowledge (Ball, Hill, & Bass, 2005; Ball, Lubienski & Mewborn, 2001; Bezgovšek Vodušek & Lipovec, 2014; Kutaka et al., 2017). The reasons for this maybe lie in elementary teachers' beliefs regarding mathematics (Leder, Pehkonen, & Törner, 2006) or also in a lack of mathematics preparation in teacher trainee programs (Taylor-Buckner, 2014). From last perspective arises the question of whether a departmentalized system - a system, in which students have a different teacher for each subject - should be established in order to improve MKT of teachers. There are known many advantages of departmentalization (see Taylor-Buckner, 2014). For example, such system allows teachers to be experts in their fields; it prevents teachers from teaching subjects where they do not feel comfortable and competent; pupils can benefit from exposure to multiple teachers throughout the day; it prepares pupils for transition to later grades, etc. Additionally, as Reys and Fennell (2003) state, "Expecting elementary teachers to have specialized knowledge in mathematics, as well as in every other subject they teach, simply is unrealistic". But on the other hand, there are also some advantages of so-called self-contained instruction system (system, where one teacher teaches almost all school subjects): individualization, more flexibility in use of time, correlation of knowledge and skills across subject, more opportunities to guide and support students' emotional and psychological development, etc. (see Taylor-Buckner, 2014).

Some research shows a positive correlation between departmentalization system in elementary schools and pupils' achievement, while there are also research, which does not confirm this relation. For example, Wright Williams (2009) has found that there is significant difference in mathematics achievement of traditional, self-contained 5th graders as compared to departmentalized system of teaching 5th -grade pupils. The differences are in the favour of the departmentalized setting. Yearwood (2011) conducted a study, very similar to that of Wright Williams and also rejected the hypothesis that there is no statistically significant difference in students' 5th grade mean math achievement, based on organizational structure (traditional vs. departmentalized) even when math scores are used as a covariate. The findings suggest that students, who participate in departmentalized system, score higher in math. Similarly, Watts (2012) considered the question whether there are differences in mathematical achievement of 4th graders who participate in departmentalized system and those who attend self-contained instruction system. The results showed that there are no significant differences. Simlarly Baroody (2017) on the basis on empirical data obtained on a sample of more than 400 classes in USA report that classroom format is not a significant predictor of achievement in mathematics.

Although increasing pupils' achievement may be one motivating factor for schools to use departmentalized classroom formats, there is a huge amount of literature emphasising importance of self-contained instruction system from psychological point of view. In just one of those studies, Chang and colleagues (2008) report that departmentalization is negatively associated with close teacher–student relationship quality, which is of highest importance in education. One of the primary objections to departmentalization is the belief that "the young child needs the one-teacher plan in order to meet his emotional-social needs" and the fear that departmentalization "leads to subject-centered rather than child-centered teaching and that it destroys the unity of the child's instructional program" (Heathers, 1969, p. 560 in Webel et al., 2016). Webel and colleagues warn, that much of "these arguments are largely based on mere opinion and speculation, revealing the need for empirical investigations of departmentalization in the elementary grades" (Webel, 2016, p.199).

As mentioned above, in Slovenia self-contained instruction system is applied, where class teachers teach almost all school subjects. All student teachers in Slovenia pursue a Master's Degree. Primary teachers, who will teach five- to 11-year-old pupils, are educated separately from Secondary school teachers, who will teach 12- to 18-year-old pupils. Students enrol in teacher education programmes after eight years of primary education and four years of secondary education. At Faculty of Education in Maribor, there are no didactic courses associated with school subjects in the first two years of primary teacher education. In the second year, students take an obligatory mathematics course and in the 3rd and 4th years, there are obligatory mathematics didactics courses. The first mathematics teaching experience involving pupils in school taught by a teacher trainee takes place in the framework of didactics of mathematics in the 3rd year, and the second in the 4th year. The students design their own lesson scenarios under the guidance of a lecturer as a part of the preparation for their teaching performances. In the 5th year, student teachers teach under the supervision of experienced in-service teachers without guidance from faculty lecturers.

As we described, during the education Slovenian student teachers gain general pedagogical knowledge and MKT. While for some of them, the general pedagogical knowledge is more important and are more focused on it than on subject specific knowledge during school training and after it, for the others the situation is inversed. In 2013 Lipovec, Podgoršek Mesarec and Antolin Drešar (2013) analysed reflections of elementary student teachers in Slovenia regarding Wiener (1986) locus of control and so-called pedagogical or subject specific focus. They reported that 51% of 4th year student teachers exhibited an external locus of control and 59% of reflections were focused on general pedagogy principles.

Research question

Our goal is to determine whether Slovenian elementary student teachers in 2019 still reflect mostly on general pedagogy aspects or the shift to mathematics pedagogy aspects of their teaching performance has occurred.

Method

Since reflections are considered to be an important part of professional growth (Schon, 1996), students teachers' reflections have been selected as primary source of data. The database was composed of the reflection (and corresponding teaching scenario) written by the student teacher at the end of the 4th year at Faculty of Education, University of Maribor. Instructions for writing reflection were relatively open; the only hint given was to reflect on the teaching performance, with word count limited to one A4 page. The reflections obtained were on average written on half of an A4 page (app. 300 words).

The sample

The sample consisted of 205 students - teachers in the 4th year of studies in academic years 2016/17, 2017/18 and 2018/19 in the Elementary Education programme at the Faculty of Education in Maribor. There were 11 (5%) male participants; the rest of the student teachers were females.

Data Analysis

The focus of reflection was determined through multiple readings by the authors. Reflection was considered to have mathematics pedagogy aspects, if in the reflection student teachers mentioned some mathematics specific elements. Student teachers could for instance mention taught mathematical content, some methodical steps in developing mathematical concepts, pupils' questions, cognitive conflicts, used mathematic specific materials etc. in relation to mathematical topics. The focus was considered to be on general pedagogy, if only general pedagogy topics could be found in a reflection. Such reflections did not provide an information regarding the school subject (in our case mathematics) of a performed lesson. Two illustrative examples follow; names of the student teachers are anonymized. Student teachers perform lessons in pairs.

Zala wrote a reflection on her geometry lesson in 2nd grade.

> Our main problem was in overrating pupils. Since we believed pupils will master the content easily we turned too much of our attention to ourselves. Nevertheless, this was an excellent experience,

as I am aware that I must strengthen my determination and that sometimes it is necessary to adopt a different teaching approach. I do not approve yelling at the pupil, because I believe that the teacher has to set up an authority in the class without raising a voice constantly. I was obviously “too kind” for this class, and this was seen by pupils as an opportunity to "enjoy” the time in my class on their own way. I think this lesson was a good indicator of what still needs to be done to make my teaching career as successful and stress less as possible (it sounds very stressful to yell every day for 7 hours). In this reflection, I focused on the shortcomings of the lesson and on the possible improvements, but I still believe that the lesson was successfully implemented, since the students cooperated and quickly understood the substance under consideration. At the end of the lesson, we did not have enough time to work out a workbook, which, however, does not seem to me wrong, since students already solved the worksheet and already consolidated their knowledge through several team activities. In order to complete the entire lesson, students should be given time constraints. For example, pupils needed a full 5 minutes, for example, to glue the learning sheets (all of them first cropped) into notebooks. Since we did not want to leave it for later, we waited for the majority to finish this assignment before we continued. This seems to me as a real school situation, because when you are a teacher and you have your class, you have to actually wait for students to crop and glue worksheets in the notebooks.

Tina reflected on her arithmetic lesson in 4th grade.

I am satisfied with the lesson. I would change the fact that in the introductory motivation, below the table sheet, where the two columns were, I would write on the blackboard both possibilities: a) the sum of ones is less than ten or b) the sum of ones is more than ten. We dictated this text, which did not prove to be the best, since some pupils have problems with dictation, especially a pupil, who needed additional stimulus. Regarding the introductory problem task, it is positive that we took enough time for it and that we waited for all the pupils to come to share their findings.. When dealing with a new learning substance, a minor error arose. It would be better to say "9 ones and 5 ones are 14 ones and 14 ones is 1 ten and 5 ones", instead of "9 ones and 5 ones are 14 ones because 14 ones is 1 ten and 4 ones". When the pupils solved the new learning material, we kept checking the results on each of them, so that all the correct records were kept when we went on to work. From the work sheets of the pre-knowledge test, we saw pupils understood the substance in the majority. One student had a difficulty in moving over ten. Three pupils were having problems with writing one number below the other. The three-digit number was moved forward in thousands, so the sum was wrong. One student was writing the digits, which should be counted on below the line where the result is, although it is usually written above the line, between the digits. Due to insufficient time, we left out the tasks in a workbook, because it seemed to us more important to give them more guided examples of the substance in question. This allowed us to come to the final analysis and find out whether the students achieved the goals of the lesson or not. Concerning the number pattern problem task, it was a bit harder to analyze, since pupils used different strategies. Most often, pupils used the same numbers for the verification of the hypothesis. Some pupils decided to randomly select the numbers they added up. Two students showed that they worked systematically, from the smallest to the largest number, as we did. To sum up, this was one of the most difficult lesson performances so far, but in the end, it was successful.

The data arising from the reflections was compared to data obtained from accompanying lecture scenarios. The lecture scenarios described in detail the course of the lesson, the objectives and methods of work. In the scenarios, the pupils' and the student teacher's activities are described, as well as the pupils' possible responses. The lecture scenarios were designed with the help of a lecturer in the faculty. The student teachers prepared the draft, and the lecturer advised the student on improving the scenario. The class teacher, who advised the student how to take into account the specifics of the class, additionally reviewed the scenarios.

Results

Table 1 shows the results of student teachers' reflection analysis.

Student teacher generation	N	Mathematic		General	
		f	f%	f	f%
2016/17	71	37	52.1	34	47.9
2017/18	85	49	57.6	36	42.4
2018/19	49	36	73.4	13	26.6
total	205	122	59.5	83	40.5

Table 1: Mathematics pedagogy reflection and general pedagogy reflection.

We would like to emphasize once more that reflections in type General did not include ANY reference to mathematics. Results show that 40.5% of student class teachers did not reflect on mathematics aspects in any way. They did not mention the covered content, pupils' mathematics misconceptions, used materials (like Dienes cubes or Cuisenaire rods), vertical trajectory or connected pre-existing mathematical knowledge, mathematical representations… Reflections of those student class teachers revolved around effectiveness of motivations, time planning, pupils with special needs, but without any reference to underlying mathematics.

There were no statistically significant differences on 5 % significance level among student teacher generations (Kruskal Wallis H = 5.670, P = 0.059). Type of student teacher reflection was also contrasted to achievement at course Didactics of mathematics II. Achievement in this course is defined with several assignments: written test (60 %), readings in geometry and algebra (15 %), work with chosen pupil on measurement concept (10 %) and digital competence in early numbers and operations (15 %). Assignments changed a bit with generations. We found no statistically significant differences (Levene F = 0.567, P = 0.452, t= -0.140, P = 0.889) in course grade regarding type of reflection.

Discussion

In reflection narratives on self-performed mathematical lesson, mathematics specific pedagogy was present on average in 59.5% of future class teachers. Additionally, inclusion of mathematics pedagogy aspects was not related to grade in mathematics education course. The share of student teachers, who reflected on mathematics pedagogy aspects decreased with generations comparing to study in 2013. In 2013, Lipovec, Podgoršek Mesarec, & Antolin Drešar reported that 26% of 4th year student teachers in generation 2011/12 does not reflect on mathematical goals at all. The situation has not improved. In last ten years, the whole team on Faculty of Education in Maribor has tried to find ways in order to achieve a "teacher change", i.e. change in students' beliefs and perceptions of learning and teaching mathematics (e.g. Lipovec & Pangrčič, 2008; Lipovec & Antolin Drešar,2014). Obviously, the effort was mostly in vain, since student teachers still do not even notice mathematics as an integral part of mathematics lesson.

Wittmann (2003) raised similar concern. He argued that German primary schools should adopt subject/mathematical pedagogical model instead of general pedagogical model in order to overcome lack of interest for mathematics as a study field. Wittman (2003), aware of possible shortcomings when embracing to mathematics oriented teaching model, states that

> …general educational principles will by no means be repealed, on the contrary: general pedagogical ideas stand namely not only in accordance with more to mathematics oriented school model, but their practical implementation becomes possible only when one considers them in framework of such a model. (Wittman, 2003, p 2)

Ochieng'Ong'ondo and Borg (2011) found that even secondary language student teachers in Kenya received more general than subject-specific pedagogy-oriented feedback from teacher educators after teaching practice, which limited the extent to which student teachers developed pedagogical reasoning. This backs up our findings.

Since research shows that a lot of elementary teachers have inadequate mathematical knowledge (Ball, Hill, & Bass, 2005; Bezgovšek Vodušek & Lipovec, 2014; Hart et al., 2013) and our results reveal that student teachers from University of Maribor still focus mainly on general pedagogy, we think, that some system changes in education of elementary teachers are necessary. Since MKT is significantly related to students achievement gains and quality of instructions (Hill, Rowan, Ball, 2005; Hill et al., 2008), some authors propose improving of the instructions by improving teachers MKT (Hill, Rowan, Ball, 2005), which may be realized thought teachers trainee programs.

Even though our study offers interesting and important results in the field of educating primary teachers as mathematics teachers, we would like to point out that it is limited to the Slovenian context, participants were almost all female, narratives were in written form and each reflection was binary coded for either mathematics or general pedagogy. Further research is needed in order to establish whether reflections differ regarding type of the lesson (new content vs. knowledge consolidation). Additionally, it is worth checking whether reflections would change if the form of handing it out changes. Now reflections are submitted to Moodle together with teaching scenarios. Sometimes both, scenario and reflection are even in the same file.

Conclusions

Despite the extensive literature about elementary organizational structures (departmentalized or self-contained), there is limited empirically based studies examining grade-level constructs. We believe that our study adds to previous research findings regarding the mathematical pedagogical knowledge of elementary student teachers. Our results show that many Slovenian student teachers emphasize the importance of general pedagogy over mathematics pedagogy. Regarding this results, we think that it is not enough that (obviously necessary) changes are limited on university courses concerning mathematics, but should be wider.

In authors opinion changes should be designed with great care not to hinder psychological and social development of young children, therefore we propose step by step procedure for introducing EMS certificate in legislation. Perhaps first the certificate should be introduced for teachers teaching in second triad of elementary school (4^{th} and 5^{th} grade). Additionally, a consideration should be made on a state level, if, on a short run, introducing a special licence for elementary mathematics specialist (EMS) is an option. Namely, it seems that programs for educating an EMS profile positively influence EMSs' development and change in significant ways (Swars et al., 2018). The main goal of introducing these changes would be to deepen the mathematical competence of Slovenian youth. In Slovenia, for now, state plans do not include initiatives for changing the education, demanded for teaching in lower grades, in any way. However, much emphasis is given on professional education. Many lower grades teachers have recognized the demand for additional education in the area of teaching mathematics and are voluntary joining courses offered by all universities and by National Educational Institute Slovenia. Until 2022, data will be collected via project focusing on in-service teachers' professional development in science and mathematical literacy (NA-MA POTI, 2017). More than 80 elementary schools and kindergartens are taking part in the project. The authors believe that evaluation of the activities will provide enough data to form an empirically based opinion regarding teaching competences of lower elementary teachers in the field of mathematics.

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Problem posing based on outcomes

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Newly reformed Serbian curriculum defines outcomes as a basis for planning instructions. The paper focusses on tasks designed to assess accomplishment of an outcome. Based on theory of representations we designed sets of four matching representational contexts of tasks corresponding for particular outcomes (symbolic, verbal using math language, realistic, and pictorial) with total of 24 tasks. A sample of 153 fourteen years old pupils (8th grade elementary school) was taken from 6 schools in Serbia. The main goal of the study was to investigate whether representational context of tasks (symbolic, math verbal, realistic verbal or iconic) results in different achievement. The analysis of the student's work reveals to what extent the representation of the problem by which we evaluate student achievement of outcomes alters the results. The findings implicate some important considerations for task designers and math instructions.

Keywords: Outcome, problem posing, representation, assessment.

Introduction

An alternative title to our paper could be *Effects of representational context of a mathematical task on student achievement of a learning outcome. This title is much longer, but it clearly states the objective of our work.* Our study was partially motivated by recent changes in Serbian school system. The ongoing educational reforms in Serbia have multiple objectives and directions. Here are some major points of changes in the school system:

- The concept of inclusive education is adopted and it is already implemented in many schools.
- There are many programs aimed at embedding technology and digital tools in teaching, learning and assessment practices.
- The Quality Assurance System is developed.
- Serbia adopted an outcomes-based approach to teaching and learning.

Outcome-based education (OBE)

The new curriculum is an operationalization of the idea of the outcome-based education (Figure 1).

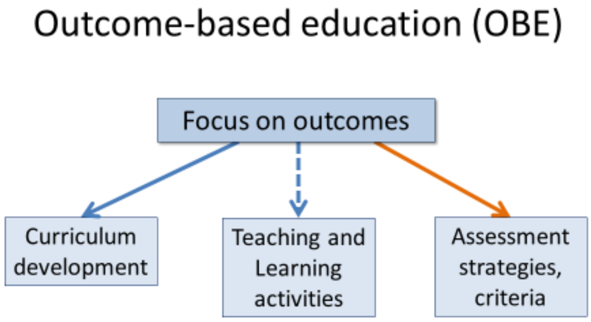


Figure 1: Key features of the reform of educational system in Serbia

The new math curriculum is defined as a set of learning outcomes that defines the content to be learned. The relationship between the learning outcomes and the content is a source of debate among many math teachers. Of course, this is expected since the education philosophy should be shifted from traditional content-based education. Unfortunately, much less attention is paid to Teaching and Learning activities, as well as Assessment criteria and strategies which are also subordinate to learning outcomes.

Our study concerns the relationship between the learning outcomes and assessment criteria and strategies. If we want to achieve substantial improvements, the strength of this link is of a key importance.

Expected benefits of the outcome based education implemented in new curriculum are: clarity, flexibility, comparison, and involvement (Ewel, 2008). The focus on outcomes makes a clear expectation of what students should learn by the end of a course, a grade level, or in a four year grade span. Students should apprehend what is expected from them. Finally, teachers should be able to structure their lessons around the student's needs. The advantage of outcome based education (OBE) is that the achievements can be evaluated and compared across different institutions. Finally, a key part of OBE is student involvement in the classroom. Ewel also pointed to some possible drawbacks of OBE: definition, assessment problems, generality and fractionation (Ibid). Outcomes could be interpreted differently. By outlining specific outcomes, a holistic approach to learning is lost. The ability to use and apply the knowledge in different ways may not be the focus of the assessment. Lastly, assessing general outcomes such as creativity, responsibility, and flexibility can become problematic. Due to the nature of specific outcomes, OBE may actually work against its ideals of serving and creating individuals that have achieved many outcomes. Of course, the traditional content-based education also suffers from many of these disadvantages.

Theory of multiple representations in teaching and learning mathematics

The theory of multiple representations in teaching and learning mathematics can help us to avoid many of the previous problems, especially assessment problems. Dealing with multiple representations and their connections plays a key role for learners to build up conceptual knowledge in mathematics. This theory is very important and it has been developed and improved by many authors.

In the last decades many research studies focused on the topic of multiple representations and their role in learning mathematics. Representations are commonly understood in mathematics community as productions that stand for embody mathematical ideas and relations (Leman, 2014). There is a broad consensus in the scientific community that dealing with multiple representations for pupils in the mathematics classroom is a highly relevant matter in learning (Janvier, 1987; Presmeg, 1986; Steinbring, 1991; Vergnaud, 1987; von Glasersfeld, 1991). Hitt (Hitt, 2002) pointed out that representations are fundamental aspects of students' construction of concepts and problem solving processes. Along the line goes contribution of others who maintained and further explored the idea that representations are tools in thinking (Bruner, 1966, Janvier & Dufour, 1987; Kaput, 1987; Meira, 2002, Cobb et al., 1992; Lesh, et al. 1987; Cuoaco & Curcio, 2001; Michalewicz & Fogel, 2000, Goldin and Shteingold, 2001). The overall idea is that full understanding of a math concept implies recognition of the relationships among different representations of the concept. Bruner's EIS principle has been utilized for early mathematics instructions. He identified three representations of concepts: inactive (action based), iconic (image based) and symbolic (language based) (Bruner, 1966). Lesh's representational system presents links between pictures, (spoken) symbols, manipulative aids, real world situations and written symbols as was studied in the context of development of the concept of fraction (Lesh, 1987). Knowledge of representations is particularly important in problem solving (Polya, 1973; Goldin & Shteingold, 2001). It appears highly relevant to help students develop abilities in making connections and conversions between different representations and in dealing flexibly with such multiple representations (Schnotz, 2014). The Serbian reformed curriculum like many others, points out the significance of dealing with multiple representations for learning mathematics.

The process of problem posing is process of purposefully combining context, math content and task format taking into account pupils' competences and experience, task purpose and other didactics influencing factors. Any problem space can be described in terms of its' context, of givens and unknown elements and of the relations between the elements (Milinkovic, 2015). Milinkovic envisioned a strategy for problem posing grounded in the theory of representations. It is important to have variations of tasks based on different representations in mathematics instructions as a method to help children make connections and develop flexibility and ability to recognize mathematical concepts presented in different forms. Our study is linked to Niem's study of the effects of changes problem formats on students' achievements. In this study students were asked to represent their conceptual knowledge in several different task contexts and formats, and performance was compared across tasks. The level of representational knowledge predicted performance on problem solving, justification, and explanation tasks. It was found that the assessments provided diagnostic information on the level and quality of individual students' understanding.

The relationship between Learning Outcomes and Assessment Tasks is in the focus of our study. We examine the influence of task representation on levels of achievement for a particular learning outcome.

Methodology

In accordance with new curriculum we were seeking to understand the impact of different representations of tasks on effectiveness of assessing particular educational outcome.

Total of 153 students from 7 classes in 4 elementary schools in Belgrade were in the sample.

In this paper first we focus attention on description of development of the instrument. Second, we analyze average scores obtained on the tests with regard to the representation of the task and alternatively to the outcome evaluated by the task.

Let's consider the following outcome statement taken from the new 8th grade math curriculum**. An 8th grade student (fourteen-year-old) is expected** to be able to solve:

- linear equations, linear inequations and systems of linear equations in two variables
- real-world problems using linear equations.

Students solve the equations from the beginning of their education. In eighth grade, they are expected to be highly efficient and strategic in solving equations in one variable. Thus, we focused on tasks designed to assess accomplishment of this outcome. We used four different representations based on the same equation. Symbolic representation (Example 1) is very typical in Serbian schools. The symbolic task is simple: Solve the given equation. In verbal representation (Example 2), we use language, words and phrases to describe mathematical ideas, connecting informal and formal levels of math language. This task can be described as a non-narrative word problem.Contextual representation (Example 3) means that mathematical ideas are put into realistic (real-world or imaginary) situations. This task can be called a story problem. Finally, in visual representation (Example 4) we use pictures, diagrams, graphs, and other drawings to represent math ideas.

Example 1 $13x + 2= 2x + 4$

Example 2 Three times a number increased by 2 equals two times the number increased by 4. Find the number.

Example 3 Philip is on holiday in Zedland, a country that uses zeds as its currency. He asked two car-rental agencies for their rental prices. The agency A rents cars at 2 zeds plus 3 zeds per kilometer travelled. The agency B rents cars at 3 zeds plus 2 zeds per kilometer travelled. For how many kilometers will Philip pay the same amount of money regardless of whether he hires a car from the agency A or from the agency B?

Example 4

The goal of our analysis was to investigate whether representational context of tasks results in different achievement.

We designed six groups of tasks, each of which has four representational variants. In total, we have 24 tasks. As exemplars we present three groups of tasks.

Group I

The underlying equation in the first group is a simple two-step linear equation with one variable. Much younger students are familiar with such an equation.

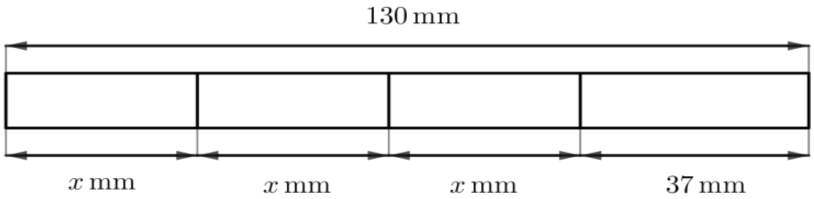
S	Solve the equation. $30x + 37 = 130$
M	If the product of a number and 30 is increased by 37, the result is 130. Find the number.
R	Marina paid 130rsd for 30 sheets of paper and a pencil. If one pencil costs 37rsd, what is the price of one sheet of paper?
V	Look at the picture below and find x/10. 

Table 1: Set of tasks based on a simple two-step linear equation

Group II

In the second group of tasks, the underlining equation is similar as in the previous one. The only difference is in the order of operations. We will see later, the M-task was a little bit problematic.

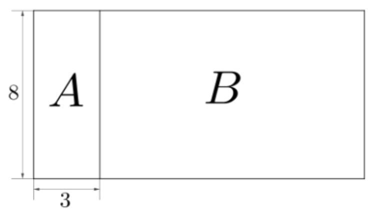
S	Solve the equation. $8 \cdot (x + 3) = 120$
M	The product of 8, and a number increased by 3, is 120. What is the number?
R	A book has 120 pages. Sofia read the eighth of the book each day. Neda read 3 pages less than Sofia every day. How many days will it take Neda to finish reading the same book?
V	The sum of the areas of two rectangles below, A and B is 120. Find the dimensions of these rectangles. 

Table 2: Set of tasks based on an equation

Group III

The third set of tasks is based on linear equations with one variable, but on both sides. Such an equation appears in 8th grade curriculum for the first time. Also, the topic of *linear functions* and their interpretations is new to 8th grade students.

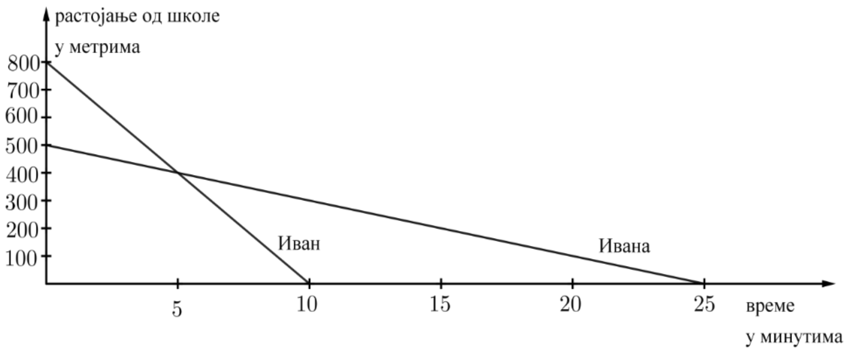
S	Solve the equation. $800 - 80t = 500 - 20t$
M	If the product of 80 and t is subtracted from 800, the result is equal to the sum of 500 and the product of -20 and t. Find the number.
R	Ivan and Ivana are classmates. They went to the school at the same time. After t minutes ($t \leq 10$), the distance between Ivan and the school is $800 - 80t$ meters. After t minutes ($t \leq 25$), the distance between Ivana and the school is $500 - 20t$ meters. How many minutes have passed since their distances from the school are the same?
V	Ivan and Ivanka are classmates. They went to the school at the same time. The graphs below show how the distance between Ivan and the school and the distance between Ivanka and the school change in time. How many minutes have passed since their distances from the school are the same? 

Table 3: Set of tasks based on linear equation with one variable

Results and discussion

We recombined all tasks, and created six different tests for students. Each test is a mix of different representations and different underlying objectives. At the bottom of the page, it is given how many students took each test. We created 6 variants based on planned combination of tasks/representations (Figure 2).

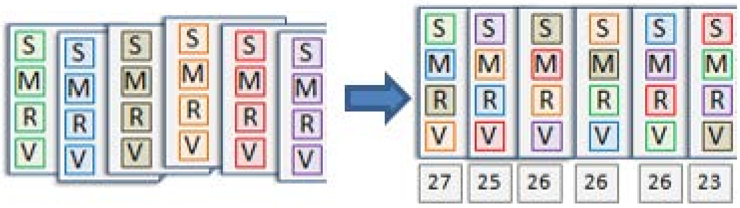


Figure 2 Combinatorics of test design, 6 groups of 4 tasks

Tasks were mixed so that each pupil solved a set of tasks such that each task had different representation and assessing different outcome. Each group of tasks (with four variants) refers to a specific curriculum defined outcome, as viewed through four (basic) representations:

Symbolic (S): which underlies the immediate application of learned procedures, mainly at the level of reproduction

Mathematical (M): understanding the mathematical language and mathematical context in which some rules are applied

Realistic (R): applying knowledge and skills in a realistic context

Visual (V): picture, diagram, graph

Instead of the traditional content-based *evaluation*, we evaluated the main steps of the Mathematical modeling process. Given the complexity of the test structure we developed specific criteria of the evaluation. Aspects of problem solution under consideration were specified for each task representation. In the case of symbolic representation of task pupils could earn from 1 to 3 points (explicit outcome content-3 points implicit content of outcomes (2). (Figure 3)

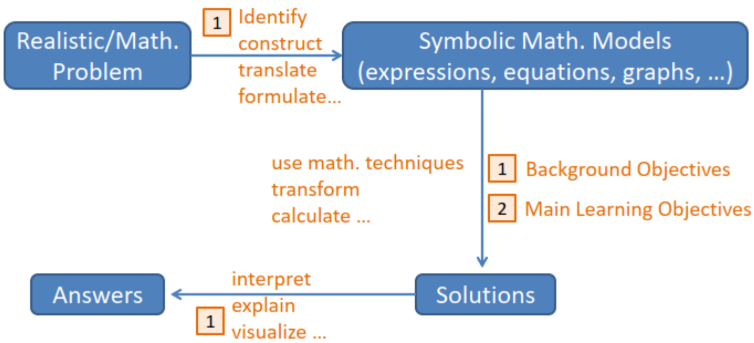


Figure 3 Assessment criteria for tasks

In our evaluation, each task was scored with max 5 points. A symbolic reformulation of the problem gives 1 point. Mathematical techniques were scored with 3 points: 1 point for background knowledge and skills, and 2 points for target objectives. The relationship between background knowledge and the (main) target learning objective is a delicate question that deserves special attention. A clear explanation of the solution that is the correct interpretation of the solution (a complete answer to the problem) is worth 1 point.

Of course, for symbolic representations, there are only two items that can be evaluated. So, we changed distribution of points, accordingly.

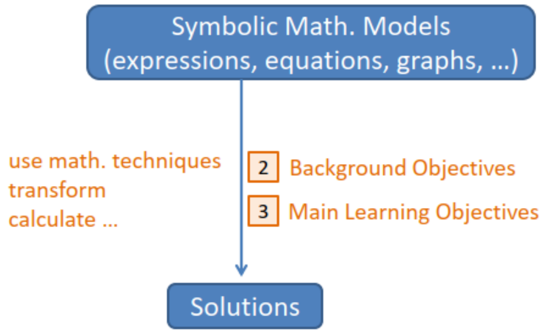


Figure 4 Assessment criteria for tasks with visual representation

The data in Table 4 presents the average scores for all tasks in the three groups presented in this paper. Note that 5 points was maximum score for each task.

	Group I	Group II	Group III
S	4.74	3.81	3.96
M	3.83	2.80	2.92
R	3.54	3.32	0.60
V	2.96	2.70	2.00

Table 4: Average scores for three groups of tasks

8th grade students are familiar with this kind of equations. We believe that the equation was not solved by those students who were not motivated to solve our test at all. But the downward arrow could be a matter of concern. The average scores of the realistic and visual representations are surprisingly low. This situation is almost the same for all other groups, and makes the general picture characteristic for all groups. Some slight exceptions were found in the Group II and Group III. That is why we have chosen these groups to present.

The verbal representation in the Group II was problematic because the students were confused what should be increased, the product or a number. The text in Serbian does not contain commas, because the students were expected to know a grammar rule that the word 'increased' refers to the last noun. It is interesting that in the fourth task, some students added a metric unit (centimeter) to each number, although it was not mentioned in the text.

The Group III tasks proved to be the most problematic for many students. In the verbal representation, there was a similar problem as in the previous group. The left side of the underlying equation 800 – 80*t (eight hundred minus eighty times t)* is verbalized as such that 'the product of 80 and *t*' is first mentioned. Thus, some students wrote and solved wrong equation.

It appears that the students were completely confused with the R-task. Each inequality in brackets is just a condition that determines an admissible domain for variable t, i.e. when the expression for distance is positive. Students didn't understand that, and they tried to combine these inequalities with the expressions, they wrote some silly inequations etc.

Are the results expected?

The chart in Figures 5 and 6 summarize average scores obtained on the test. If we reorder bars, and calculate the arithmetic mean for each representation, we obtain a picture that opens many interesting topics for a discussion.

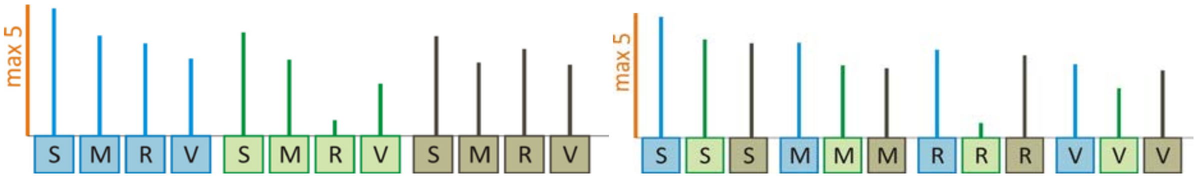


Figure 5 The arithmetic mean for each task

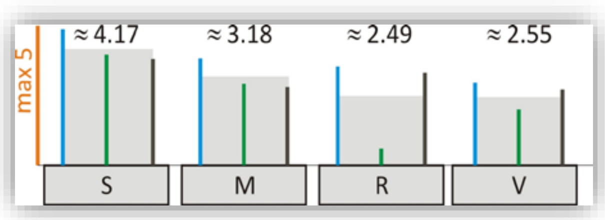


Figure 6 The arithmetic mean for each representation

For example, it is well-known that visual representations are powerful way to understand abstract mathematical ideas. Visual representations and relaying on realistic context are staring points in teaching and learning mathematics. Why they become stranger than mathematical and symbolic representations?

We predicted that students would have more difficulty with problems presented as stories or non-narrative word problems than with those presented as symbol equations. But it is surprising for us, that students struggled with visual representations. It is possible, that our assessment criteria had a significant influence on our results.

Conclusions

We don't have definite, good answers to many questions arising in our study. We believe that our research has certainly shown at least two things. First, the theory of multiple representations must be seriously taken into account in development of assessment strategies. Second, assessment criteria should be aligned with the mathematical modeling cycle. The first item approves the importance of theories that research in mathematics education gives us. The second item emphasizes the importance of the mathematical model process and mathematical way of thinking. These are the main reasons why we are given so much time to teach mathematics.

Appendix

Background objectives vs. Main Learning Objectives

Find the product of 12, 7.5 and 11.2, and then determine whether it is greater than or equal to 1000.

A water tank is in the form of a cuboid (rectangular parallelepiped). Its dimensions are 12dm, 7,5dm and 11,2dm. Can the tank contain 1000 liters of water (1 l = 1dm^3)? Explain your answer.

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Poetess Desanka Maksimović's high school graduation exam in Mathematics

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Hundred years between 1919 and 2019 is, by all standards, a long period, during which effort has been made to develop and improve mathematics education in Serbia. The paper investigates the resulting changes in terms of students' knowledge. The final exam in Mathematics for the year 1919, when Serbian great poetess Desanka Maksimović graduated, has been used in the empirical study. A sample of 508 high school graduation students (aged 18-19) participated in the study. The paper presents results of the analysis in details. Comparison with the achievements of the generation 1919 is used as a reliable basis for a number of commentaries and, in our opinion, several valuable conclusions, including some proposals concerning the curricular reforms.

Keywords: Reasoning, formal knowledge, "anti-normal" distribution, retention of knowledge.

Through the mist
of unclear words I feel
the beauty of a song.

Dogs remember for a long time
benefaction made to them.
Man forgets it.

Desanka Maksimović

Introduction

Serbian great poetess Desanka Maksimović was born on May 3, 1898 (according to the "old" calendar that is on the May 16, according to the "new" one) in the village of Rabrovica (district of Valjevo) as the daughter of Draginja and Mihailo Maksimović (a teacher in Brankovina, famous for his poetical skill). She graduated from the Valjevo Gymnasium exactly a hundred years ago, in September 1919. From the documentation, carefully conserved in the Valjevo Grammar school in the book "The Main protocol of maturation examination from the school year 1918/1919" (Fig.1); under the number 8, DesankaMaksimović, graduating student was registered. From the mentioned Protocol it can be seen that she completed the graduation exam consisting of 7 subjects: Serbian, Latin, French, Mathematics, History general and national, including Geography, Chemistry with Physics, and Jestastvenica (a mixture of Biology and Geology).

Figure 1, **left. The main protocol of the maturation examination**

Figure 2 right. Page 13 from the records of the examination

Mathematics is, undoubtedly, a part of the general culture of mankind. However, everybody of us could testify about, that a lot of people, belonging to the circle of approved cultural creators (literates, actors, musicians, …) are boasting with the fact that he (she) did not know or disliked mathematics. Desanka Maksimović knew and loved mathematics. Based on the excellently completed written part of the final examination in mathematics, she was released of the oral part and the final grade was excellent. Let us bear witness, V. Andrić personally, as a pupil, professor and director of the Valjevo High School, that, in the frame of her public appearances, poetry evenings and conversations in her school, she used to express her love and admiration about the beauty and significance of mathematics and its close relations with the poetry. Desanka Maksimović's written part of the final exam in mathematics took part on September 17th, 1919. According to the records of the examination (page 13) two groups of problems were offered (at student's choice) (Fig.2). Translation of the problems from the protocol is given in Figure 3.

FIRST GROUP

1. Someone sells certain goods with a gain of 15% and earns a profit of 66,70 RSD. How much revenue would have been generated if the profit was 20%?

2. Solve the equtions system
$x^2 + y^2 = 20$, $\log x + \log y = 0{,}90309$.

3. The base of a cast-iron prism has a shape of equilateral triangle with side $2m$, and the height of the prism is equal to the doubled height of the basic triangle. What is the weight of this prism if the specific weight of the cast-iron is $7{,}2 kg/m^3$.

SECOND GROUP

1. Two amounts are to be paid, one in year, another, exceeding the first for 10000 dinars, in 15 months. Esconting with 4.5% is giving total escont of 3600 dinars. What are the amounts?

2. Solve the equations system:
$5\sqrt{x} - 3\sqrt{y} = 11$, $4\sqrt{x} - \frac{1}{2}\sqrt{y} = 24$.

3. Given a cube of edge a. What is the difference of volumes of the circumscribed cylinder and the inscribed cylinder of this cube?

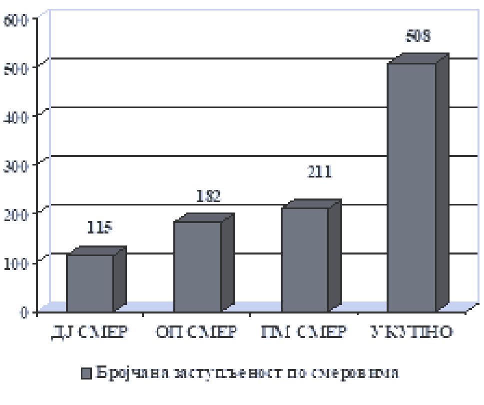
Figure 3 Translation of the original tasks

"Mathematics curriculum comes from cultural context and therefore it cannot be looked upon independently from culture" (Milinkovic, 2018, p. 141). Any curricular reform means to reshape, to make different always with intention to obtain better results at the end. "But mere change does not mean improvement" (Schubert, 20 p.80). Our research is dealing with (slightly modified) problems from year 1919 and the question: How the facts are standing a hundred years later? The aim of modification was primarily to make problems comprehensive for pupils in 2019.

Methodology

The survey was carried out in March 2019. All involved schools and colleagues received: texts of tasks; instructions for the realization of the research; key for reviewing and evaluating tasks; a form for statistical processing of results. The solving of tasks was managed by colleagues – professors of mathematics from ten gymnasiums in Serbia or longtime collaborators of the research authors. The pupils were doing a school-class experimental task (45 minutes); in addition to controlling tasks, colleagues performed a review of the tasks and statistical processing of the data, obtained from the tasks review. Solutions to all tasks were scored according to the unique key that all implementers of the research had got in advance. The essential purpose of the key is to eliminate, as much as possible, the subjective factor of the reviewers' access, to attain equally scoring criteria for each individual task and to equate the evaluation of every work as a whole. The scoring scale had the following, more or less standard, intervals: 0-29 – inadequate (mark 1); 30-49 – enough (2); 50-69 – good (3); 70-85 – very good (4); 86-100 – excellent (5).

The sample of 508 pupils from 10 grammar schools (8 cities, 20 classes) in Serbia was included in the research. That makes about 3.2% of the total number of all graduates in Serbia in the school year 2018/2019. The sample consists of 5 classes of social-linguistic course (SL), 7 classes of general course (G) and 8 classes of natural-mathematical course (NM), (Fig.4 and Fig. 5).



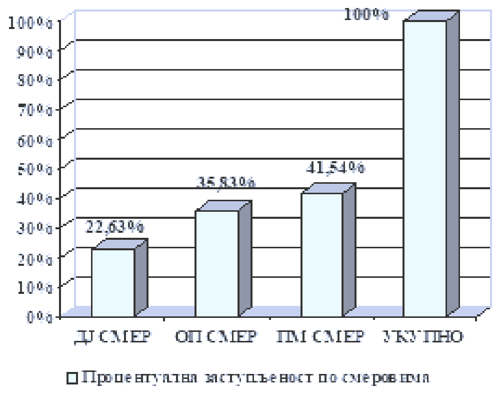


Figure 4. Number of students per course

Figure 5. Percentage of students per course

The cumulative research results are presented in two detailed (the total 100 of points is divided into 10 equal subintervals) bar graphs, representing the distribution of the number of pupils as well as their relative frequency, achieving the corresponding scores (number of points) (Fig.6).

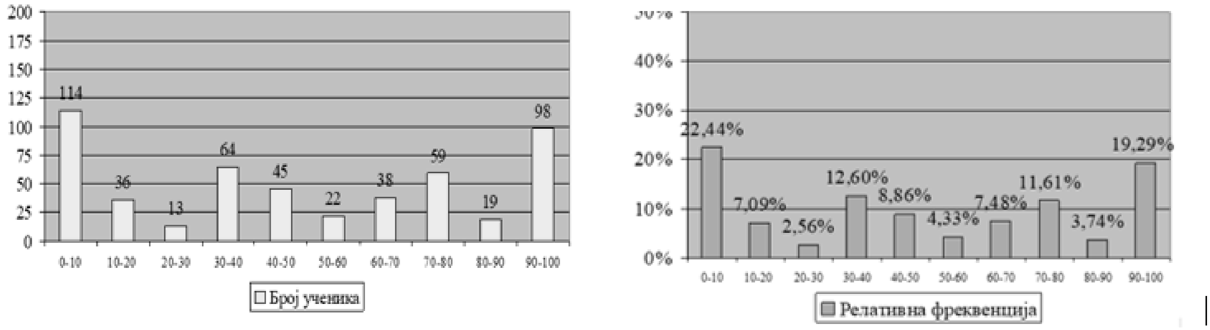


Figure 6 Number of pupils (left) and relative frequency (right)

Analysis and preliminary commentaries

Descriptive statistical analysis. By direct calculations we obtain the following characteristics of scores: absolute arithmetic mean 46.52; weighted arithmetic mean 48.68; median 46.00; absolute mean deviation26.69; its standard deviation 33.362. From the distribution of scores one can notice: the most frequent is the interval from 0 to 10 points (114 pupils - 22.4%), including a disappointing fact that the majority of them (even 94 pupils - 18.50%) had zero points (regardless of whether this is a result of their ignorance or low motivation). Second in frequency is the interval from 90 to 100 points (98 pupils – 19.29%); this is an excellent data, especially if one takes into account the fact that even 68 pupils (13.40%) had the maximum score of 100 points. From the bar graphs on the Fig.9 it is obvious that the realized distribution of pupil's scores is unexpected; instead of the expected normal distribution we obtained a picture that we can characterize as an "*anti-normal*" distribution; so, in contrary to the usual case of making the outermost intervals of the outcomes the least frequent, they appear the most frequently.

In our abstract announced goal of comparing the results in solving the same, a hundred years "old", problems by the graduators in the school year 1918/1919 and those of the 2018/2019, one should agree with some limitations. The first of them is of technical character; the sample of Valjevo gymnasium graduates in 1919 is relatively small (34 members only), but we were unable to remove this restriction, while the sample of tested pupils (508) is fairly solid and can be accepted as a representative one. Another one is a consequence of the substantial distinction between the populations of pupils included in the gymnasium education in two observed periods. In the structure of the former population secondary school pupils, there were less than 10% of them attending gymnasium education while, according to the official records of the Statistical Institute of Serbia, the corresponding percentage in the school year 2018/2019 is 26%. This evidently testifies about the significantly higher elitist characteristic of gymnasiums education in the former period. Under assumption that the total mental potentials of two populations are more or less comparable, such distribution provides a kind of advance for the former population. Also, the fact that the motivation of the present generation, to participate in a research arranged by an unknown person(s), was lower, could affect their success. Finally, the research was going on in March, three month before their maturation exam, while a lot of activities were on, including preparation courses for the entrance exams at faculties. This could result by the problem of content forgetting, causing the reduction of presence of the knowledge retention.

Detailed analysis of the proposed problems

Some of the original problems from 1919th needed a slight modification, in order to make them completely clear for the current population; one of them (group II, the first problem) was replaced by another problem, from the same topic, because the concept of esconting is unknown for them. The formulation of problem is followed by bar graphs, showing the number (left graph) and percentage (right graph) of pupils whose solution was: correct, partly correct, incorrect, did not try to solve.

The first problem

(I) Someone sells certain goods with a gain of 15% and earns a profit of 66.70 RSD. How much revenue would have been earned if the profit was 20%? (30 points)

(II) Three house painters paint five apartments in four days. a) How many apartments will 24 painters paint in 7 days? b) How many house painters will paint 30 apartments in 9 days? c) For how many days will 12 painters paint 50 apartments? (30 points)

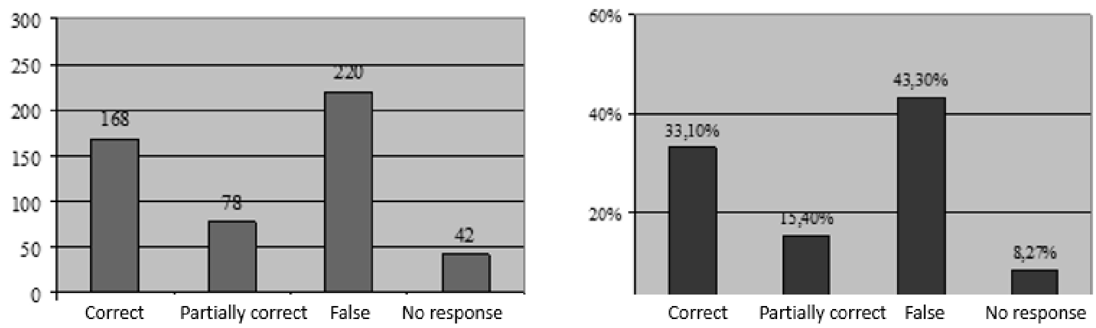


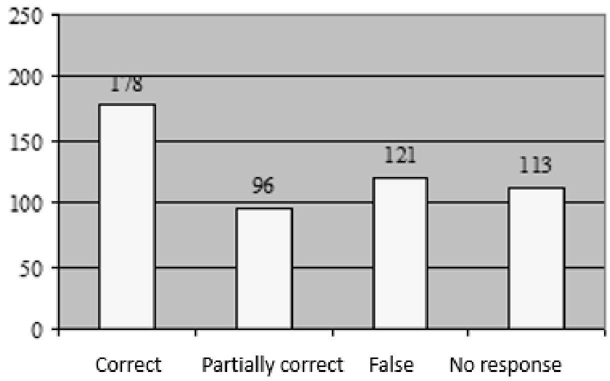
Figure 7 Pupils' achievement on Problem 1

The proportionality of magnitude was completely surmounted by about one third of the tested pupils. Over 50% of students have shown that they are unable to solve any problem of income, nor a problem of "painting". Further analysis of the papers will make us unambiguously convinced that they are able to resolve the problem (proportion) formally, but also that often the setting of proportion has been incorrect. Also, one can observe that the pupils' critical attitude toward the obtained results is poor enough and that their inquiry regarding the reality of the obtained result does not exist. For example, the wrong conclusion that "… several house painters in several days commit painting of smaller number of apartments then fewer painters in fewer days …" is present in some of the presented "solutions". All this suggests that the thoughtful activity of the current generation is rather weak, mostly formal and mechanical, even in the situation when they are facing with a usual activity of the daily life. From the aspect of retention of knowledge, the process of forgetting in such situations cannot exist, because various proportionality problems appear in making many important decisions, connected with the everyday life. The lack of mathematical culture and necessary apparatus in this area can be harmful for the existence and functioning of the family, the progress of the company, the assessment of business move, etc.

The second problem

(I) If, approximately, $log\ 2 = 0{,}30103$, solve the system of equations
$x^2 + y^2 = 20$, $log\ x + log\ y = 0{,}90309$. (40 points)

(II) Solve the system of equations
$$5\sqrt{x} - 3\sqrt{y} = 11, \quad 4\sqrt{x} - \frac{1}{2}\sqrt{y} = 24 .$$ (40 points)



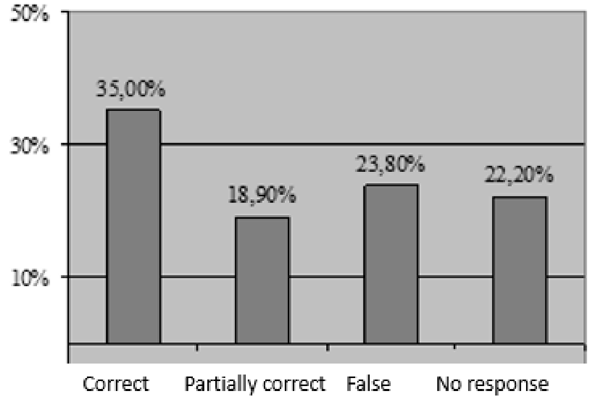


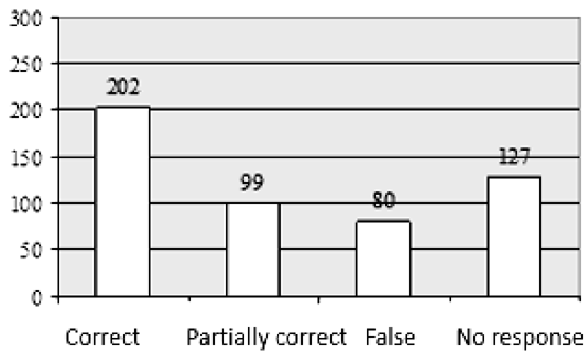
Figure 8 Pupils' achievement on Problem 3

Systems of equations were, to a large extent, mastered by over half of the tested pupils; the percentage of incorrect solutions equals 46%. The system of linear equations is solved better; those who have partially correctly solved the system (18.90%) mainly fall into negligence in solving the system of logarithmic equations, where points are rejected in solutions where there is a missing solution, or where there are negative solutions. This task shows that the secondary school graduates do orient themselves much better in the formal field and that their technique is a minor problem in comparison with the fundamental understanding of matter.

The third problem

(I) The basis of a cast-iron prism has shape of an equilateral triangle with side of $2m$, and the height of the prism is equal to the doubled height of the basic triangle. What is the weight of this prism if the specific weight of the cast-iron is $7200 kg/m^3$?

(II) Given a cube of edge a. What is the difference between the volumes of the circumscribed cylinder and the inscribed cylinder of this cube?



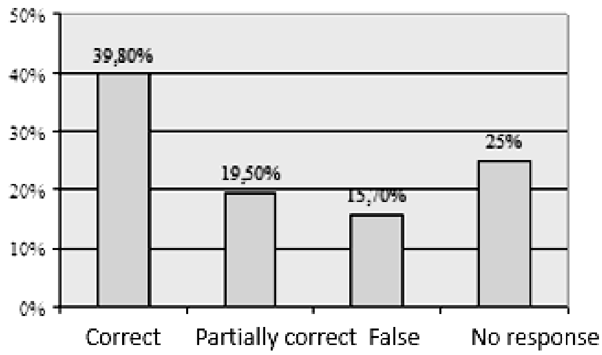


Figure 9 Pupils' achievement on Problem 3

This problem has been completely solved by close to 40% of tested pupils, and unsolved by a little more than 40% of them. The task is composed of several steps, including the component of application, and the attained results are satisfactory; let us list some explanations, arguing in favor of this. An acceptable explanation is the fact that the acquaintance with the formulae for the area of equilateral triangle, volume of the prism and of the cylinder, relation between the weight, specific weight and the volume of a solid, that means formal knowledge, leads directly to the solution without any major misunderstandings. In addition, we should take into account that the stereometry was treated recently, in the previous third class. A more detailed analysis of the papers shows that errors in solving this problem are mostly originated by poor skills in computing or ignorance of formulas.

Unified discussion of the problems

The third, stereometrical problem is best solved, while the worst solved, first problem is related to the proportionality of magnitudes. In all three problems, the percentage (and number) of the inaccurately solved and unresolved problems is higher than the percentage of accurately resolved problems (Figure 10). The very noticeable difference is present in the case of the first problem (18.47%) in comparison with 11% for the second problem and irrelevant 1% for the third one. Taking into account all the tasks cumulatively, 36% of them had been solved completely, another 17.80% partially (with certain omissions or numerical errors); that makes a total of 53.80%. On the other hand, 27.80% of problems were recorded as inaccurately solved, while even 18.40% were completely unresolved (because of the participators complete ignorance or some other reasons); that makes a total of 46.20% unsatisfactory managed tasks (Figure 11).

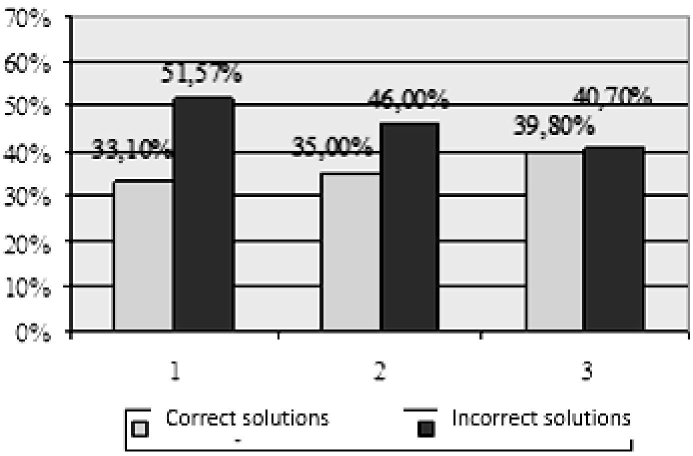


Figure 10 Three tasks

Observing the totality of three problems, we can notice that 36% of problems were correctly solved. By adding 17.80% of partially correct solutions, we obtain that 53.80% of problems were solved. There have been 27.80% of false solved problems; by adding 18% of problems with no response, one gets' the total of 46.20% unsolved problems. The last information is extremely disturbing.

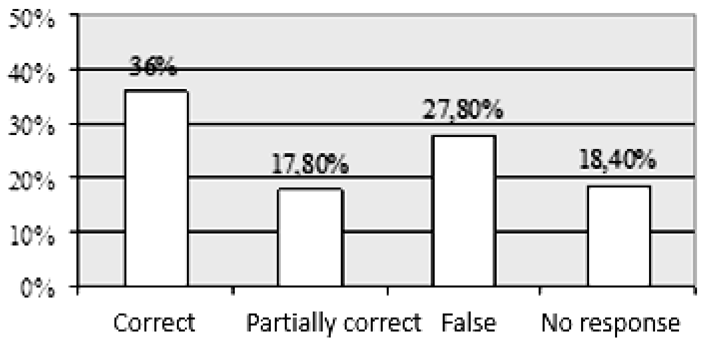


Figure 11 Total score on the test in 2019.

If those of the graduating pupils from the school year 2018/2019 who have solved three, two, one or none of the problems would be marked as excellent, very good, good or unsatisfactory respectively, according to the classification rule from the school year 1918/1919, then it can be observed that in the 2018/2019 generation of graduating pupils there are slightly more excellent (2%), almost equal very good, but significantly more (10%) good, and unsatisfactory (7%) works.

General results and final analysis

Let us remind the scoring scale of the pupils' achievement from the school year 1918/1919, consisting of four levels (with numerical valuation): excellent (5), very good (4), good (3), failing

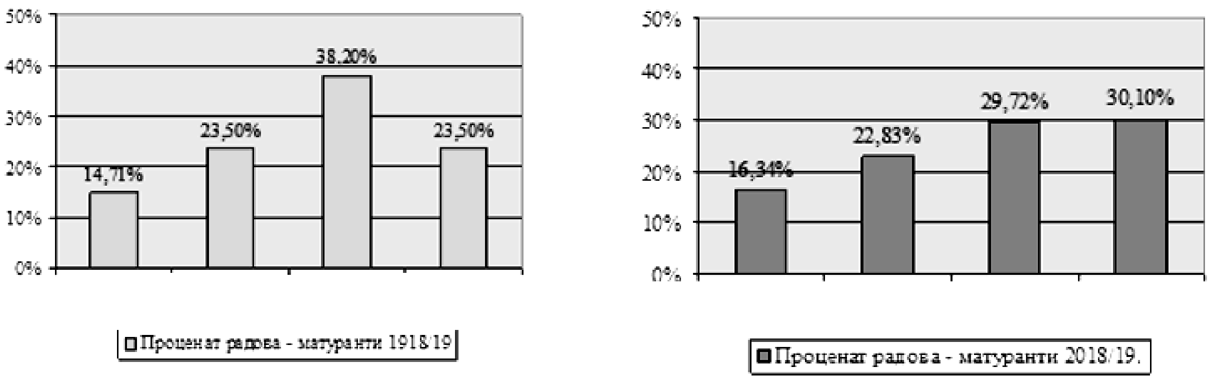


Figure 12 Achievements of Generation 1918/1919 and Generation 2018/2019

(2), and the actual scoring scale, mentioned in the frame of the *Methodology*. Two bar graphs of the Figure 12 can be used (under some conditions) for a kind of direct comparison of two populations, we are dealing with.

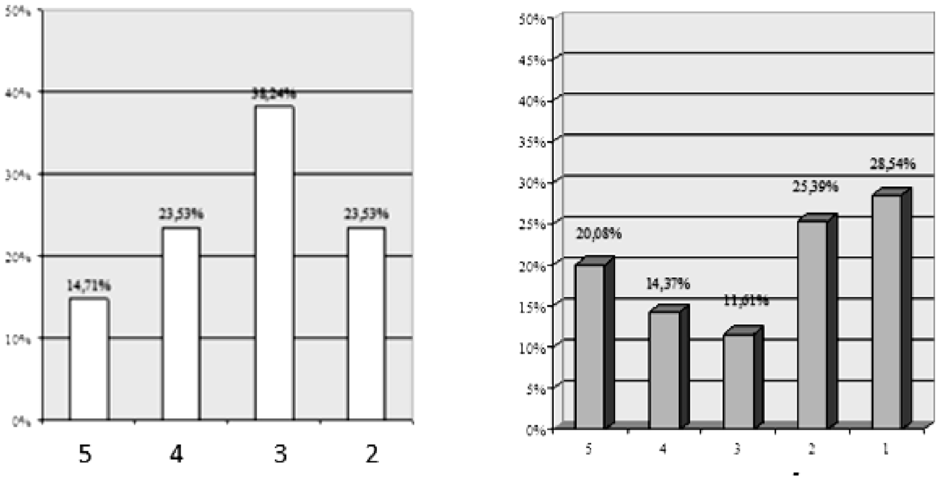


Figure 13 Comparative graphical presentations of achievements of Generation 1918/1919 and Generation 2018/2019

It is evident that the former generations' success is somewhat better; this conclusion follows directly from the fact that the percentage of pupils attaining positive success is 76.47%, while the same indicator for the 2018/2019 generation is 71.46%. Even more convincingly in favor of this assertion, by direct comparison, one can notice that total of 76% of excellent, very good and good pupils in the former generation is significantly larger than the corresponding total for the generation 2018/2019, making 46%. The difference in the scoring scales requires an additional discussion about the 25.39% of pupils from the generation 2018/2019, graded with the mark 2 (enough); their attainment is poor, close to failure and is not encouraging. We are convinced that the mean score is more likely to arbitrate in the desired comparison. For the tested generation it is 2.72, while for the former generation it is 3.06 or 3.18, depending on the treatment of pupils whose score has been graded with mark 2.

The distribution of the success for the generation 1918/1919 can be qualified as nearly normal, while the corresponding distribution for the generation 2018/2019 shows an "*anti-normal*" (our term) tendency, since the peripheral features (grade 5 on one and grades 1, 2 on the other side) are far more frequent then grades 3, 4, expected to be the most frequent in the normal distribution.

In the frame of these considerations it would be interesting to analyze the success in courses, though it seems that for each course the sample is rather unrepresentative. Therefore, the derived conclusions should not be accepted with somewhat higher degree of reliance.

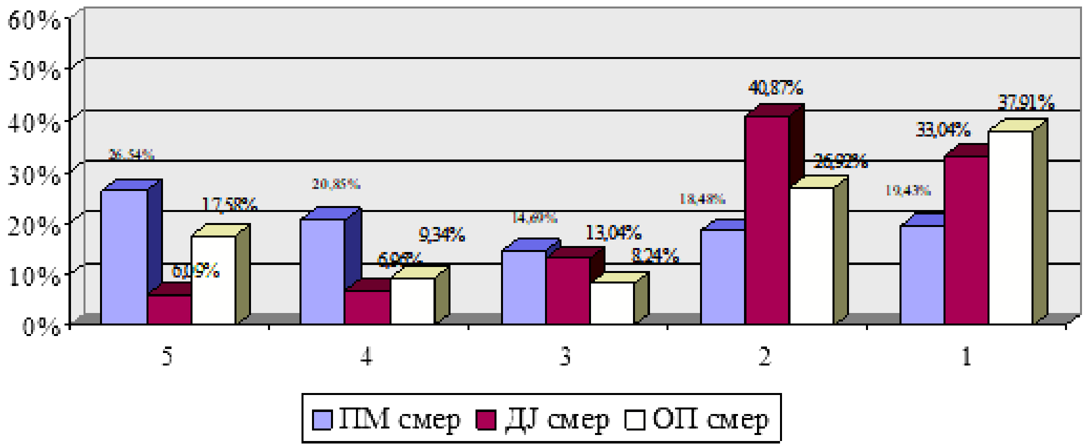


Figure 14 Achievements by courses ПМ - Natural sciences and Mathematical, ДЈ - Social sciences and Linguistic, ОП - General

If judged by the percentage of insufficient test scores, the best success is achieved in the science-mathematical course, and the worst in the general course. If as a criterion we take the average grade of success, it is highest in the sciences-mathematical course again (3.17), followed by the general course (2.41) and the social-linguistic course (2.12).

In all three courses the mentioned "anti-normal" distribution of the success' of pupils' that had participate in the analyzed test can be recognized.

Conclusions and one suggestion

Observing the results of the presented research on the whole, with the achievements of the 1918/1919 generation as the control sample, and taking into account our remarks concerning all the distinctions between the two, a hundred years "distant", generations, we are ready to make a number of conclusions (C) and a suggestion (S).

C_1 – We are satisfied with the fact that about 20% of tested pupils had achieved an excellent result, including the detail that more than 13% had attained maximum of 100 points;
C_2 – We are not satisfied with the results of tested pupils because about 31% of them did not solve any task, and about 29% did not get passing score;
C_3 – Pupils are much more successful with the formal tasks than with the tasks requiring creative reasoning;
C_4 – The recognized "anti-normal" distribution of the pupils' achievements demands complex and serious investigations, including a wide circle of participants from the education system and besides;
C_5 – The obtained results indicate also that the quality of knowledge, duration of its functional usability, even its retention, essentially depends on the "distance" of the relevant contents in the curriculum.

S – In the processes of curriculum reforms in mathematics education, all the participants should be obliged to take into account some of the above listed conclusions. All professional mathematicians and educators, involved in preparing curricular materials for the decision makers, should attend to hierarchical nature of mathematical knowledge but also to point to fundamental knowledge elements which need to be repeatedly activated during school years in order to provide retention of prolonged usability.

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Advantages and disadvantages of heuristic teaching in relation to traditional teaching

- the case of the parallelogram area -

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In this paper, the authors compare the results of two distinct teaching methods - teaching with the heuristic and elements of problem-solving approach and traditional teaching of mathematics, in particular on determining the area of the parallelogram. One group of students (experimental) had to come up with an appropriate rule for determining the area of the parallelogram through heuristic approach, with the help of manipulatives such as tangram and paper models. In the second (control) group the same teacher taught students in the teacher – centered classroom climate. The research was conducted in 2017. in Kragujevac, Serbia, with the sample of 59 students of the sixth grade. After classes, students were tested to examine possible differences in their understanding, theoretical and practical knowledge, depending on the teaching method. Students from the experimental group also expressed their impressions through the questionnaire about the classes.

Keywords: Heuristic approach, problem solving, teacher – centered classroom, parallelogram area

Introduction

The aim of this paper is to compare the mathematics teaching methods and their influence on students' learning mathematics, on one concrete case - the case of the parallelogram area.

Some of the major teaching methods are: traditional, problem-solving, and discovery learning. Traditional teaching method is a teacher-centered instruction, while problem-solving method is a teacher and student-centered which is based upon how teacher uses the four steps of problem-solving methods by Polya (1957), in teaching Mathematics. In discovery (heuristic) learning method, teacher plays the role of facilitator through involving students in varied activities associated with the discovery and construction of the knowledge. In an educational context, heuristics and problem solving are essential processes for student to learn and to self-regulate learning. On that basis we tried to explore the advantages and disadvantages of that teaching and learning approach in relation to traditional teaching and learning. In the research presented in this paper, we analyze the process of teaching and learning the determination of parallelogram area in the elementary school "Stanislav Sremčević", Kragujevac, Serbia. Data were collected using class observations, students' answers on the test they solved and students' questionnaires.

This research actually presents the pilot research that can be considered as an introduction into a much wider research of the advantages and disadvantages of heuristic teaching and traditional teaching methods and their influence on students' achievement in understanding, theoretical and practical knowledge of triangle area and quadrilateral area.

Theoretical background

Studies have shown that students experience mathematics anxiety which is a feeling of tension and fear that interfere with learning mathematics (Miller & Bichsel, 2004; Whyte, 2009). This may be in relation to the applied teaching methods in the classrooms. In order to succeed in teaching mathematics, teachers should be aware of different teaching methods and practicing those methods.

In a mathematics class using traditional method, the teacher reviews previous material and homework, and then demonstrates low-level problem solving followed by seatwork imitating the teacher's demonstration (Stonewater, 2005). This pedagogical approach of placing the primary focus on the teacher as a transmitter of knowledge (that is, teaching by telling) is representative of behaviorist theory (Hackman, 2004). The common method of teaching mathematics using traditional method is a teacher-centered and giving lecture to the students is the dominant situation in this teaching method.

On the other hand, discovery (heuristic) method is based on the constructivist approach which is a learner-centered approach that emphasizes the importance of individuals actively construct their knowledge and understanding the connections between mathematical concepts with the guidance from the teacher. In this method, teachers should not attempt to simply pour information into children's minds, but to give children confidence to discover their knowledge, consider, and think critically with supervision and significant guidance of the teacher (Eby, Herrel & Jordan, 2005). This method also, emphasizes that students should form their own interpretation of evidence and to check those interpretations with the teacher. Teachers obligation is to listen to students' thoughts and ideas and to guide them to make proper conclusions. The teacher's responsibility is also, to introduce situations and contexts within which the learner may construct appropriate knowledge. Within this kind of teaching in mathematics education, students create their own understanding of every mathematical concept, so that the main responsibility of teaching is not explaining, lecturing, or attempting to transfer mathematical knowledge, but creating situations for students to promote their mental structures. In discovery method, students are not passive recipients of the knowledge, but they construct new mathematical knowledge by reflecting on their physical and mental actions. Therefore, learning reflects a social process in which children engage in dialogue and discussion with themselves as well as others as they develop intellectually. The development of logical reasoning, the chance to work as a team, engagement and concentration are some of the attributes, which can be advanced through Mathematics, by the heuristics approach.

This approach emphasizes finding a good representation of the problem. First „to understand a problem, then, the problem solver creates (imagines) objects and relations in his head which correspond to objects and relations in the externally presented problem. These internal objects and relations are the problem solver's internal representation of the problem. Different people may create different internal representations of the same problem. Frequently, problem solvers will make an external representation of some parts of the problem. They do this by drawing sketches and diagrams or by writing down symbols or equations which correspond to parts of the internal representation. Such external representations can be enormously helpful in solving problems” (Hayes, 1989, p. 5).

In some publications, authors (Al-Fayez, Mona, Jubran & Sereen, 2012) point out that there is no difference in the students’ outcomes regarding the teaching method. Others showed that the results of the students’ learning by discovery are better than the results of students’ learning taught by expository (Kistian, Armanto & Sudrajat, 2017). Research on education reflects disagreement on the best method of teaching mathematics, but some authors claim that there are proofs that if the goal is to foster deep, conceptual understanding of mathematics, students must be provided with opportunities to develop their own ideas of the underlying concepts through exploration (Marshal & Horton, 2011). When it comes to teacher influence in this teaching method, research reveals that a reasonable degree of guidance is often more effective than low teachers’ guidance (Lazonder & Harmsen, 2016). **S**tudents’ self-evaluation of their efforts in math was the topic in some research and the conclusions can be found that their self-evaluation of their efforts is higher in the innovative learning environment than in the traditional. (Mason & Scrivani, 2004).

Methodology

For the purpose of the research, the authors prepared the scenarios for the classes that concerned determining the parallelogram area, one for the traditional approach and the other for the heuristic approach with the elements of problem-solving. The research was conducted in 2017. in Kragujevac, Serbia, with the sample of 59 students of the sixth grade. For each type of the class (with the experimental group by heuristic approach and the control group with the traditional approach), almost the same number of students (30 in the experimental and 29 in the control group) participated. The groups were formed within two six grade classes where the mean differences between their previous mathematical knowledge (expressed by their marks in mathematics) were the smallest. Students from those two classes, in the elementary school "Stanislav Sremčević“, Kragujevac, Serbia had approximately the same grade in mathematics – there was no statistically significant difference ($t = 0.029, p = 0.977, df = 57,$) between the accomplishments in mathematics of the students from the experimental group ($m = 3.44, SD = 1.36$) and the students from the control group ($m = 3.45, SD = 1.09$). Data were collected using class observations, students’ answers on the test they had after the classes in which they learned about the determining parallelogram area and questionnaires that gave authors insight in student’s impressions about the classes.

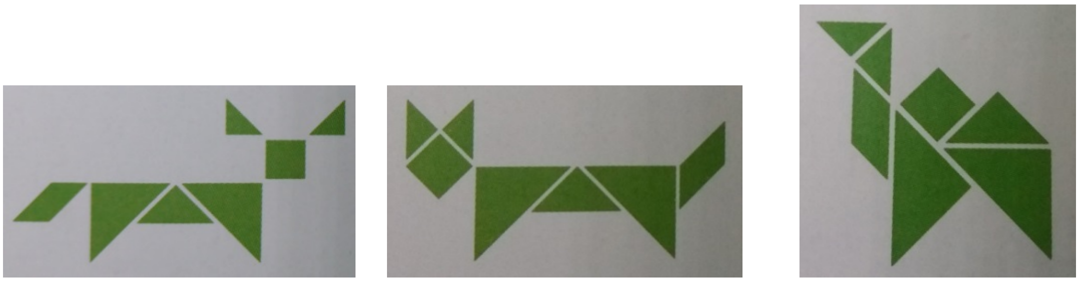
Statistical analysis was carried out by using SPSS Statistics software package. The calculations in the SPSS software were carried out at the level of significance of 0.05.

The teacher-centered approach

Within the control group, class went in the usual order. The teacher informed the students about the aim of the class, that they were up to learn how to determine the parallelogram area of the given side length and the given height of the parallelogram. Using chalk and blackboard, with the help of the geometrical tools as rulers, teacher showed students how to reorganize the parts of parallelogram to form the rectangle which area they know how to determine. After pointing out the rule for determining the parallelogram area, teacher gave simple tasks to the students where they made simple calculations with the help of the rule for determining the parallelogram area. Teacher emphasized that parallelogram area can be determined with the given side a and height h_a, but also with the elements b and h_b. First few tasks were solved by the teacher, and after that the students - volunteers went in front of the class and solved the given tasks. They calculated the parallelogram area for the given parallelogram side and height, and later determined the side/height for the given area of the parallelogram and the height/side of the parallelogram.

The heuristic approach

For the experimental group, the authors prepared the game Tangram and some problems for students to solve. In the introductory part of the class, the teacher gave to the students Tangram set, (each pair of students got one set) and depending on the schedule of students sitting, assigned them a task (each group were solving a different task: A, B or C). The task was to determine the area of the following figures (Figure 1).



Task A Task B Task C

Figure 1: Tasks for the introductory part of the class for the experimental group

It was expected that by reorganizing the given parts of the figures, students will assemble the square, (in form that the Tangram was given to the students) then measure squares side length and with the formula for calculating the square area to solve the tasks. At the end of the introductory part of the class, the teacher started a discussion which aimed to the students' conclusion that by reorganization, or by moving parts of the figure, the area is obtained no matter how the parts of the figure are organized.

After that the teacher said to the students to separate a few parts of the Tangram, to assemble the parallelogram from those parts and to determine the area of such parallelogram (Figure 2).

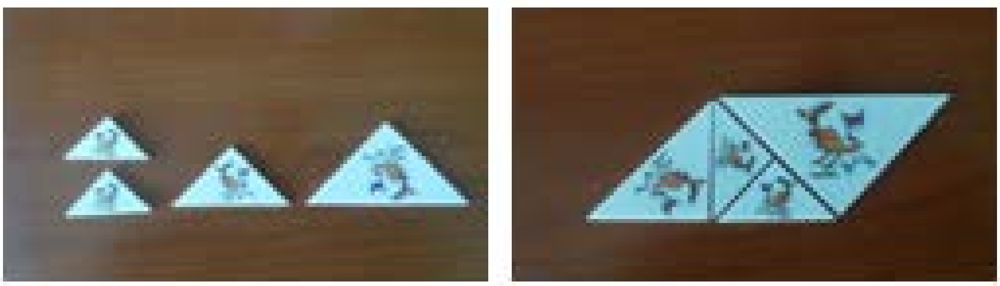


Figure 2: The parts of the Tangram set from which the parallelogram can be obtained

Over the half of the number of students (18 students) rearranged the given parts into the rectangle (Figure 3) and after the measuring the adequate lengths, determined the parallelogram area.

Figure 3: The rectangle obtained by rearranging the parts of the parallelogram

A discussion between students themselves and between students and the teacher occurred that lead to students' discovery of the rule for determining the area of the parallelogram, because rectangle and parallelograms have a pair of matching sides, and the height of the parallelogram is equal to the other side of the rectangle. The teacher then gave manipulatives to the students - the parallelogram paper models in different colors, so they pasted it in their notebooks, after drawing parallelogram heights, cutting the parallelogram models and rearranging them into the rectangle models (Figure 4). These students' activities aimed to students' understanding that parallelogram area can be determined with the given side a and height h_a, or with the elements b and h_b.

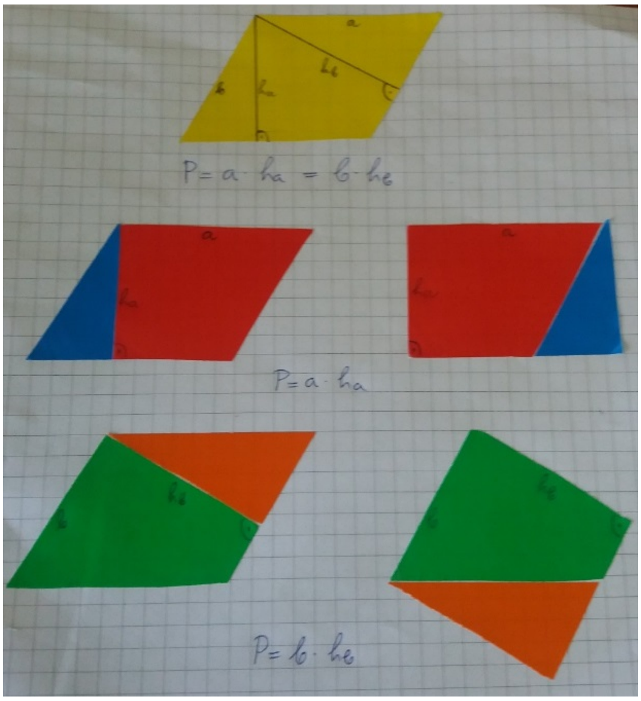


Figure 4: Pasted paper models of parallelogram in the student' notebook after drawing the parallelogram height, cutting parts of the parallelogram and rearranging them into the rectangle model

In the class where the teaching method was discovery method, a few difficulties appeared. Those difficulties were in the form of lack of precision and handling with the cutting of the corresponding figures, as well as with precise gluing of the corresponding parts. Due to the lack of time, the teacher and students have not been able to do all the planned tasks after pointing out the rule for determining the parallelogram area. More precisely, the teacher solved only one elementary task, for which the rule for determining the parallelogram area was directly applied.

The authors attended both the lectures, with the students of the Faculty of Science (at that time). The attendees were following the classes, recording the reactions, students' activity and students' interest, they were not involved in student-teacher discussions, and every one of them focused on certain number of students in order to get as more data as possible. After classes, on the first following class, students were tested and interviewed.

Results and discussion

First of all, we analyzed the students' work during the classes, their motivation, their interest and activity. In the traditional method, in the control group, more than 50% of the students participated in the discussion of the implementation of the rule for the parallelogram area, but many of them (15-20 depending on the task) avoided discussion and thinking about the tasks, reducing their activity exclusively to rewriting tasks' solution from the table. During the class, in the experimental group, almost all students (26 students) were focused on solving the problems they were given.

After that, we analyzed the students' test results. For the test (appendix), students had six tasks. Through the first three tasks students' ability to determine the parallelogram area with the given side and height were tested as well as their ability to properly measure the mentioned elements and determine the area, and to determine the other height of the parallelogram when one of them is given as well as the parallelogram sides. In the task 4 students had to show understanding of the difference between the equal parallelograms and the parallelograms with the same area and that if the parallelograms are equal then their area is also equal, but implication doesn't stand in the other way. Since in the both classes (in the experimental group and in the control group) only the case where the bottom of the height belongs to an adequate side of the parallelogram was examined, the task 5 gave us insight in what level are students capable to manage the situation where the height bottom doesn't belong to the proper side and to show that in that case the rule for determining parallelogram area stands as well. In the task 6, we examined students' capability for application the rule for determining the parallelogram area for determining area of other figures. Here (Figure 5) is given one sample test.

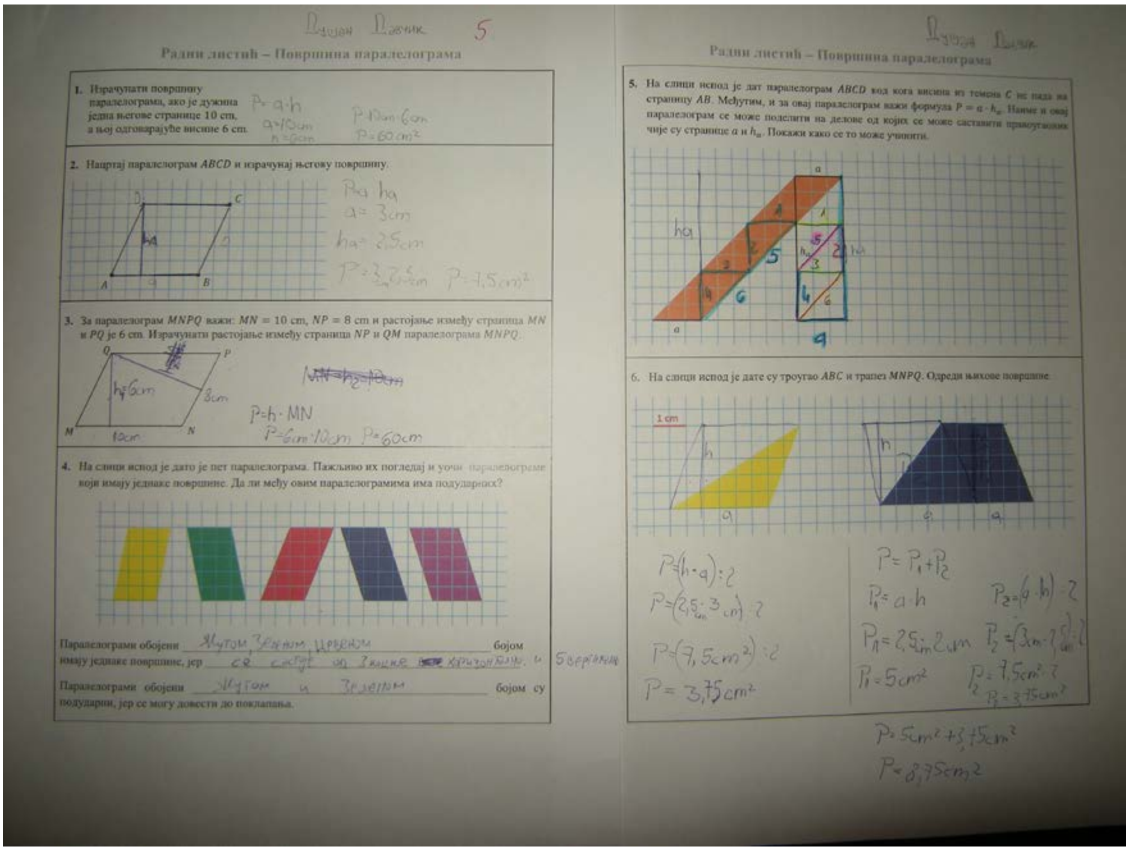
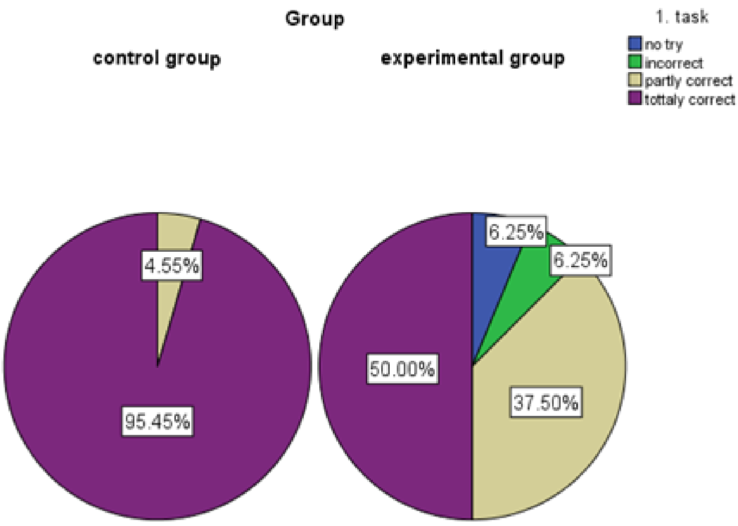


Figure 5: An example test from one student from the experimental group

For the test, the maximum number of points was 30, tasks were evaluated partially. Students' papers were analyzed, and every test example was evaluated. As we said earlier, there was no statistically significant difference ($t = 1{,}83, p = 0.075, df = 57$) between the test results of the students that were taught in the traditional manner ($m = 21.13, SD = 6.22$) and the students from the experimental group ($m = 18.75, SD = 5.49$), which learned through discovery. But there were some interesting results. When it comes to results of the first two tasks, it can be seen (Figure 6) that students exposed to teacher – centered classroom climate prevailed.



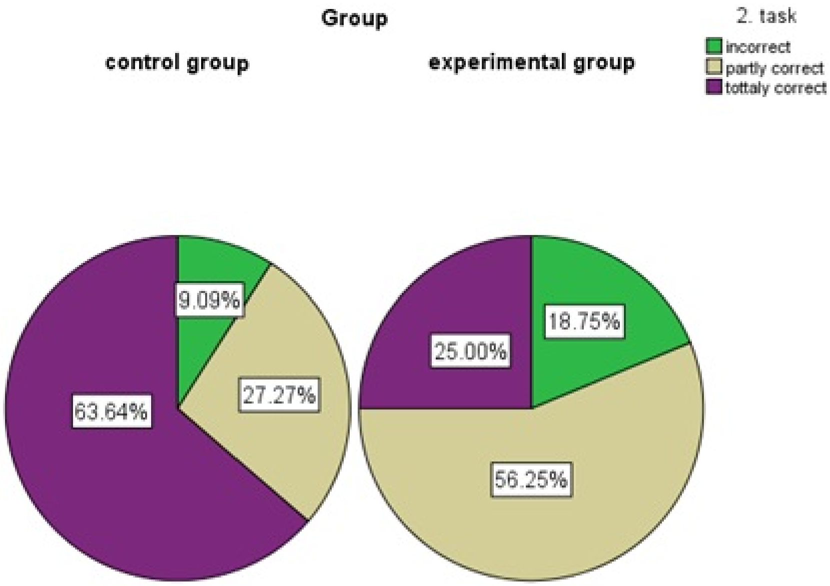


Figure 6: Frequency of solving the task 1 and task 2, depending on the group

For instance, in the task 1 on the test (the side length and the height of the parallelogram were given, see appendix) all the students from the control group tried to solve the given task and allmost all of them (95.45%) were successful, while only the half of the students from the experimental group (50%) correctly solved this simple task. In the task 2 students needed to measure the proper lengths and with the rule for determining the parallelogram area to calculate the wanted area. Almost two thirds (63, 64%) of the students taught by the teacher in the teacher-centered classroom were totally successful while only a quarter (25%) of the students taught with the discovery method came to the correct solution. As we said earlier, in the control group 9 tasks were made during the class, while in the experimental group teacher solved only one simple task. This implies that for the less demanding tasks, larger number of students taught in traditional way, give the wanted answers. Having in mind that in order to evaluate students' achievements, teachers are generally inclined to give students this kind of tasks as tasks in which almost all students should be successful. So, one of the potential lack of the heuristic approach is a doubt that students with the low accomplishments in mathematics will not be able to gain a positive outcome on such a test.

On the other hand, when it comes to more demanding task, task that includes deeper understanding of the area concept and general idea for determining area of the figure, heuristic approach exceeded the traditional teaching approach. For instance, number of students from the experimental group (21, 62%) that gave proper answer to the question "why the parallelograms have the same area" in the task 4 (Figure 7) was much larger comparing to the number of students from the control group (4,55%).

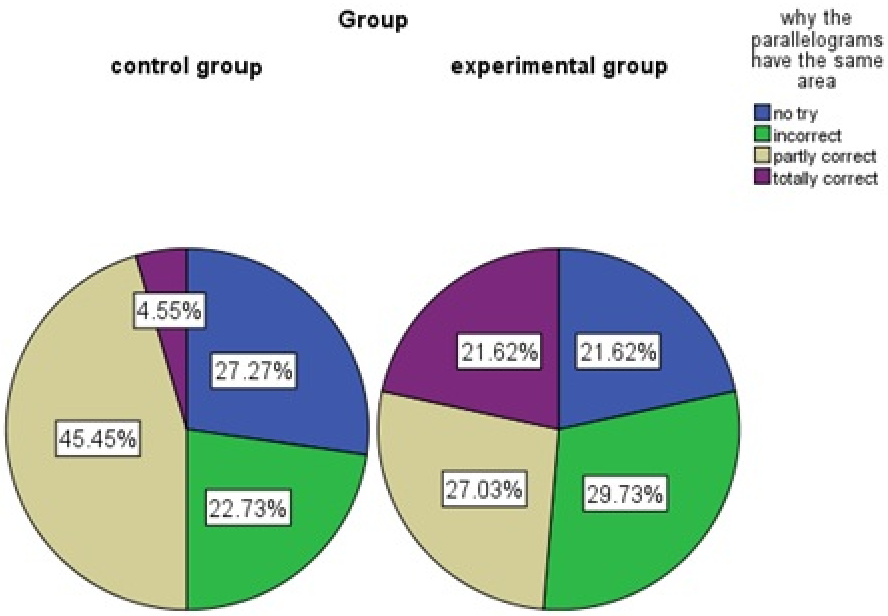


Figure 7: Analysis of students' answers about the parallelograms with the same area, depending on the group

In the task 5 we wanted to examine how would students manage the situation where the height bottom doesn't belong to the proper side (that case was not processed during the classes). As could be seen in a Figure 8, exactly 50% of the students from the experimental group solved the given task totally or partially (with some minor mistakes) correct, while only 27, 27% of the control group were in the mentioned categories.

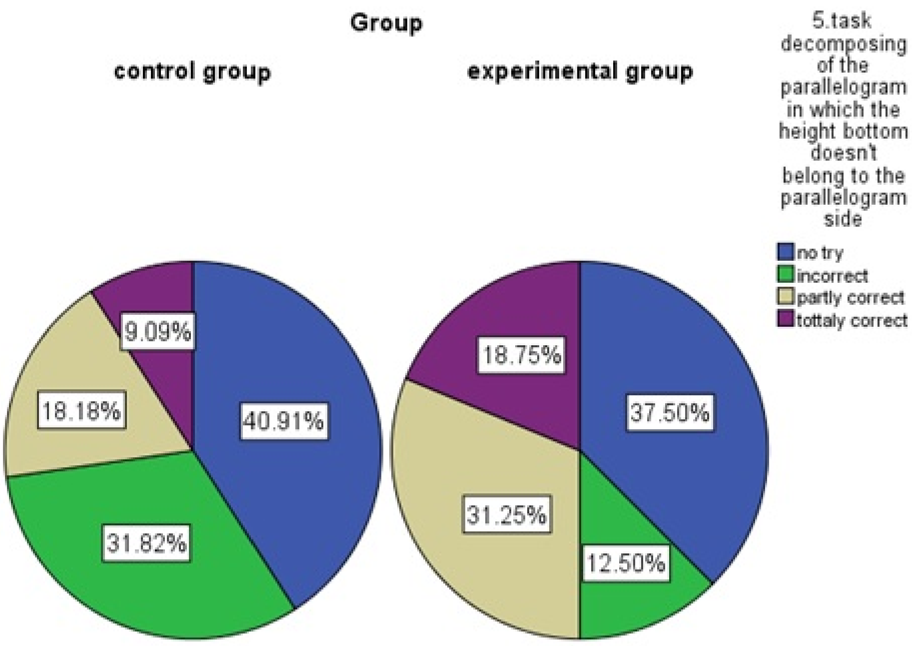


Figure 8: Analysis of students' answers about the parallelogram area where the bottom of the height of the parallelogram doesn't belong to the side of parallelogram, depending on the group

Also, in the task 6 we wanted to examine were there differences between the two groups of students in establishing the foundation for adopting the rules for determining the triangle area and trapezoid area because the two mentioned rules are derived and based on the rule for determining the parallelogram area. After the analyzing the students' results, we came to the results that there was no large difference in students' results. The first and the second part of the sixth task properly solved the same number of the students from the control group (13, 64%) while from the experimental group the first part solved 10,81% and 12,50% of them solved the second part of the given task.

As the heuristic approach is less present in the class, we asked the students from the experimental group about their impressions about the class, the teaching method and the manipulatives (Tangram and adequate parallelogram models). Through the questionnaire that students crossed after the lecture with the heuristic approach, 85% of the students said that the class was more interesting than usuall math classes. All of the students claimed that they liked using the manipulatives (Tangram and color paper) in class, and all of the interviewed students claimed that manipulatives helped them in learning rules for determining the area of the parallelogram.

Conclusion

During the classes and after, by analyzing students' behavior on both the observed classes, the authors noticed some of the advantages and disadvantages of both presented teaching approaches. The advantages of the heuristic approach of teaching mathematics for elementary level students are higher students' motivation and higher students' activity. Students that were in the control group and learned about determining the area of parallelogram in the teacher-centered atmosphere, showed better results in easier tasks, tasks in which they apply the formula for determining the parallelogram area, lectured by the teacher on the previous class. On the other hand, students who learned through discovery showed better results in the tasks that demand higher level of mental activities, so the authors assume that this kind of teaching might better develop students' mental activities. Of course, one class can't be enough for that claim, so that would be one of the research questions for the following research. While we were preparing the research we also contacted numerous colleagues (older, more experienced teachers and also younger colleagues, former students) mathematicians and asked them what were their impressions about the heuristic approach in their working experience and a few answers appeared most often: Students are generally more active and motivated for the class; It's not easy to create a problem for every math class; A hard problem can "block" the class; Smaller amount of the classical tasks can be made. Those answers mostly match authors' impressions about advantages and disadvantages of heuristic teaching in relation to traditional teaching, also with the literature and the authors are interested and curious if the following research will confirm those advantages and flaws or find some disagreements.

We have to emphasize once again, that analysis was performed only after one class where the students learning in teacher-centered classroom climate practiced more, while students who learned by discovery method revealed more, and there was no statistically significant difference between their results. The question for the broader, and much larger study is will there be differences between the methods: teacher-centered, discovery method with the problem-solving elements and the combination of those methods in the longer term for determining the triangle and quadrilateral area.

Having in mind that not all of the students made proper conclusions while learning through the discovery (which was confirmed by their results on the test) and that they did not have time for exercise and to deepen their knowledge about the determining parallelogram area, authors also assume that it is essential that this kind of class (with heuristic approach) is followed by the exercise class to achieve the desired results. The authors, on the behalf of the pilot research: on the students' activities, positive students' reactions during the class in the experimental group, test results of both the groups and the questionnaires results, feel free to recommend that teaching methods should be varied to allow more dynamical teaching that motivates students, but also to provide sufficient time for students' practice.

Acknowledgment

Part of this work was created within the subject „Research in the teaching of mathematics" at Faculty of Science, Department for mathematics and informatics, in master studies of mathematics, in 2017. The authors would like to thank the elementary school "Stanislav Sremčević", Kragujevac, specially the colleague Sanja Milojević, who realized the classes described in this paper and students of master studies of mathematics: Milica Nikolić, Marija Svetozarević, Želimirka Glintić and Nevena Nešović, who were partially involved in data collecting during the research.

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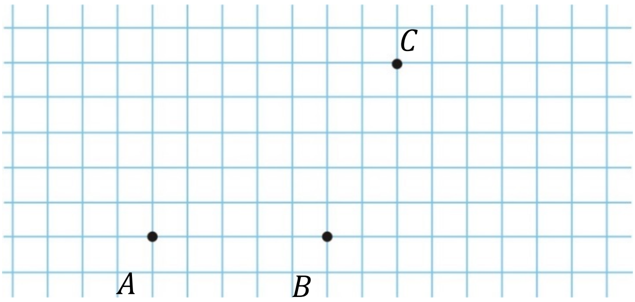
Appendix

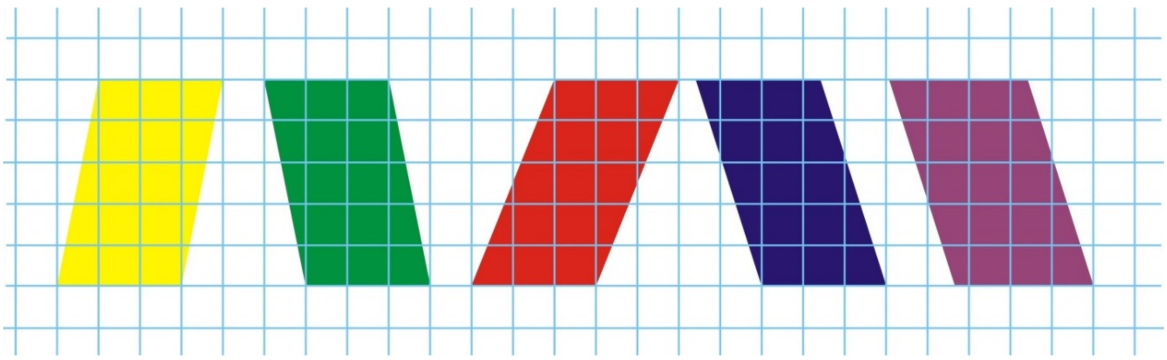
1. Determine the area of the parallelogram if the side length is 10 cm and the corresponding height of the parallelogram is 6 cm.

2. Draw the parallelogram $ABCD$ and determine the area of such parallelogram.



3. For the parallelogram $MNPQ$ side lengths are $MN = 10$ cm, $NP = 8$cm and the distance between the sides MN and PQ is 6 cm. Determine the distance between the sides NP and QM for the parallelogram $MNPQ$.

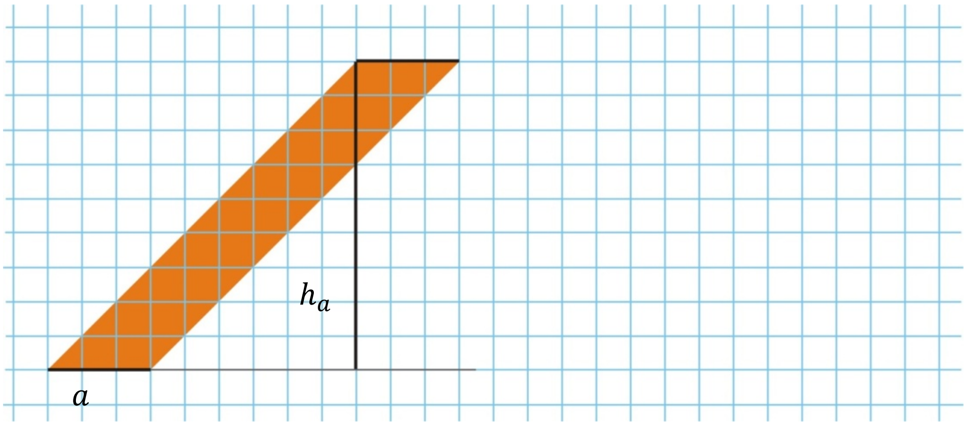
4. On the picture below five parallelograms are given. Look carefully at them and give the answers to the following questions.



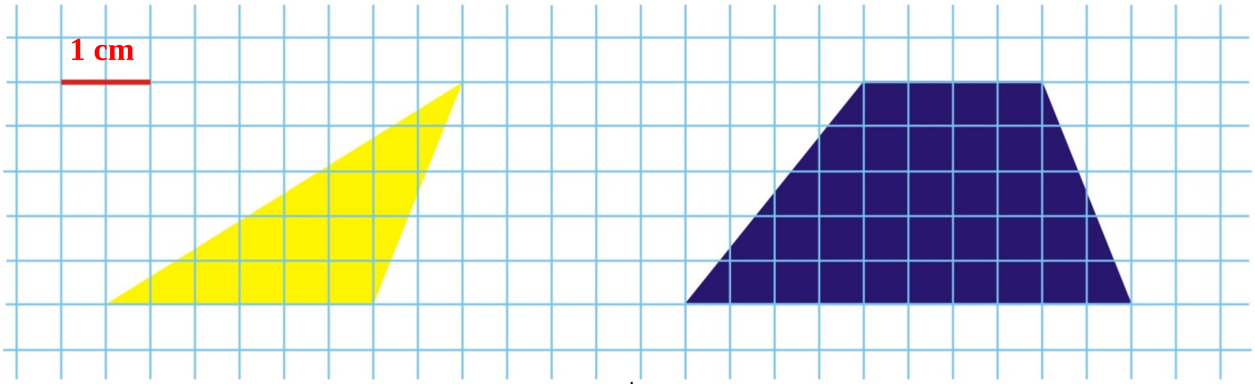
Parallelograms colored with the ____________________________________ colors have equal areas beacause ____________________________________.

Parallelograms colored with the ____ colors are equal (matching) parallelograms.

5. On the picture below parallelogram *ABCD* is given where the height from the vertice *C* doesn't belong to side *AB*. However, the formula $P = a \cdot h_a$ also applies to this parallelogram because this parallelogram can be divided into parts of which a rectangle whose sides are a and h_a can be constructed. Show how this can be done.



6. Determine the areas of the triangle and the trapezoid, given on the picture below.



A different approach to solving linear Diophantine equations. An experimental study on using multiple strategies to solve linear Diophantine equations

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The most common methods for solving the linear Diophantine equations (LDE) of form $ax + by = c,\ \ a, b, c\ \in \mathbb{Z}$ that students learn in advanced classes in primary schools are Euclid alghorithm, Euler method and solving by guessing or inspection. This paper introduces an alternative method in solving LDE of form $ax + by = c,\ \ a, b, c\ \in \mathbb{Z}$. The LDE is solved using the properties of Farey sequence and in this paper the algorithm for solving LDE is given. In the second part of the paper an experimental study is described and the answer on the following question is given. Can using multiple strategies lead to greater gain in solving LDE, or does it lead to confusion? The experimental study was conducted on 124 seventh-grade students, who had been divided in four groups. Each group of students were acquainted with different numbers of methods for solving linear Diophantine equations. Post-test results show an interesting difference in solving LDE from group to group. Also, in this paper we present the main reasons for using the specific method to solve LDE from students' point of view. One year later we conducted measurement in retained knowledge and procedural flexibility and here we present our findings.

Keywords: Research methodes, retained knowledge, procedural flexibility, linear Diophantine equations, Farey sequence.

Introduction

Each of us had at least one math teacher who taught us that the way he is solving some mathematical problem is the best way or it is the only way to solve problem and that alternative solutions will not be accepted. Usually, in that kind of classes students are deprived of opportunities to explore and to build flexible mathematical knowledge. On the other hand, there are teachers who are using multiple strategies in their classes, who motivates and encourages his students to compare, discuss, and critique multiple strategies in problem solving. In this paper by multiple strategies, we refer to students' ability to solve mathematics problems in more than one way; as noted by Silver and colleagues, "It is nearly axiomatic among those interested in mathematical problem solving as a key aspect of school mathematics that students should have experiences in which they solve problems in more than one way" (Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005, p. 288). Our experience as students as well as our pedagogical experience tells us that too much new information can lead to confusion. Confusion is often associated with reaching a cognitive impasse or "being stuck" while trying to learn something new (Woolf et al., 2009). Confusion is hypothesized to occur when there is an ongoing mismatch between incoming information and prior knowledge that cannot be resolved right away, when new information cannot be integrated into existing mental models, or when information processing is interrupted by inconsistencies in the information stream (Mandler 1984, 1990; Stein & Levine, 1991).

Solving linear Diophantine equations (LDE) is a content of advanced classes in seventh grade in Croatian primary schools. We did the research on a heterogeneous group of students, because only 22 of them had a prior knowledge in solving LDE. For that this kind of group was an ideal to make quantitative and qualitative research with a goal to get the answer on the following questions. *Can using multiple strategies lead to greater gain in solving LDE, or does it lead to confusion? What are the main reasons for using the specific method to solve LDE from students' point of view?*

A different approach in solving linear Diophantine equations

The most common methods for solving the linear LDE of the form $\mathrm{ax + by = c},\ \mathrm{a, b, c}\ \in \mathbb{Z}$ that students learn in advanced classes in primary schools are Euclid alghorithm, Euler method and solving by guessing or inspection. In experimental study we used one more, less known method for solving LDE in which the properties of Farey sequences and mediant properties are used. The Farey sequence F_n of order n is a sequence of completely reduced fractions between 0 and 1 which, when in lowest terms have denominators less than or equal to n, arranged in increasing order. Given $0 < \frac{a}{b} < \frac{c}{d}$, than fraction $\frac{a+c}{b+d}$ is called mediant of fractions $\frac{a}{b}$ and $\frac{c}{d}$. More about properties of Farey sequence and mediant you can find for example in Dujella (2011/2012) and Lončarević (2018). Kriewall in (Kriewall, 1975) showed that $x = b$ and $y = -a$ are solutions of LDE $cx + dy = 1$, where c and d are coprime natural number and $0 < c < d$. Fraction $\frac{a}{b}$ is an adjacent fraction to $\frac{c}{d}$ in Farey sequence of order $n \geq d$. This method doesn't give a correct solution if $0 < d < c$. Also, it is obvious that this method doesn't work if $c > 0$ and $d < 0$, $c < 0$ and $d > 0$, $c < 0$ and $d < 0$. Here we give an algorithm to solve LDE of form $\mathrm{ax + by = c}$, where $\mathrm{a, b, c}\ \in \mathbb{Z}$. Also, the appropriate examples are given.

Algorithm for solving LDE

a) Let $a > 0$, $b < 0$ and $c = 1$.
   A. If $a < |b|$, then we look a consecutive fraction $\frac{x}{y}$ to $\frac{a}{|b|}$, such that $\frac{x}{y} < \frac{a}{|b|}$ in Farey sequence $F_{|b|}$. If that fraction is $\frac{x_0}{y_0}$, then $x = y_0$, $y = x_0$ is a particular solution of equation, and all solutions are given by $x = y_0 - bk$, $y = x_0 + ak$, where $k \in \mathbb{Z}$.
   B. If $a > |b|$, then we look a consecutive fraction $\frac{x}{y}$ to $\frac{|b|}{a}$, such that $\frac{|b|}{a} < \frac{x}{y}$ in Farey sequence F_a. If that fraction is $\frac{x_0}{y_0}$, then $x = x_0$, $y = y_0$ is a particular solution of equation, and all solutions are given by $x = x_0 - bk$, $y = y_0 + ak$, where $k \in \mathbb{Z}$.

b) Let $a < 0$, $b > 0$ and $c = 1$.
   - If $|a| < b$, then we look a consecutive fraction $\frac{x}{y}$ to $\frac{|a|}{b}$, such that $\frac{|a|}{b} < \frac{x}{y}$ in Farey sequence F_b. If that fraction is $\frac{x_0}{y_0}$, then $x = y_0$, $y = x_0$ is a particular solution of equation, and all solutions are given by $x = y_0 - bk$, $y = x_0 + ak$, where $k \in \mathbb{Z}$.

- If $|a| > b$, then we look a consecutive fraction $\frac{x}{y}$ to $\frac{b}{|a|}$, such that $\frac{x}{y} < \frac{b}{|a|}$ in Farey sequence $F_{|a|}$. If that fraction is $\frac{x_0}{y_0}$, then $x = x_0$, $y = y_0$ is a particular solution of equation, and all solutions are given by $x = x_0 - bk$, $y = y_0 + ak$, where $k \in \mathbb{Z}$.

c) Let $a < 0$, $b < 0$ and $c = 1$.

C. If $|a| < |b|$, then we look a consecutive fraction $\frac{x}{y}$ to $\frac{|a|}{|b|}$, such that $\frac{x}{y} < \frac{|a|}{|b|}$ in Farey sequence $F_{|b|}$. If that fraction is $\frac{x_0}{y_0}$, then $x = -y_0$, $y = x_0$ is a particular solution of equation, and all solutions are given by $x = -y_0 - bk$, $y = x_0 + ak$, where $k \in \mathbb{Z}$.

D. If $|a| > |b|$, then we look a consecutive fraction $\frac{x}{y}$ to $\frac{|b|}{|a|}$, such that $\frac{x}{y} < \frac{|b|}{|a|}$ in Farey sequence $F_{|a|}$. If that fraction is $\frac{x_0}{y_0}$, then $x = x_0$, $y = -y_0$ is a particular solution of equation, and all solutions are given by $x = x_0 - bk$, $y = -y_0 + ak$, where $k \in \mathbb{Z}$.

d) Let $a > 0$, $b > 0$ and $c = 1$.

E. If $a < b$, then we look a consecutive fraction $\frac{x}{y}$ to $\frac{a}{b}$, such that $\frac{x}{y} < \frac{a}{b}$ in Farey sequence F_b. If that fraction is $\frac{x_0}{y_0}$, then $x = y_0$, $y = -x_0$ is a particular solution of equation, and all solutions are given by $x = y_0 - bk$, $y = -x_0 + ak$, where $k \in \mathbb{Z}$.

F. If $a > b$, then we look a consecutive fraction $\frac{x}{y}$ to $\frac{b}{a}$, such that $\frac{x}{y} < \frac{b}{a}$ in Farey sequence F_a. If that fraction is $\frac{x_0}{y_0}$, then $x = -x_0$, $y = y_0$ is a particular solution of equation, and all solutions are given by $x = -x_0 - bk$, $y = y_0 + ak$, where $k \in \mathbb{Z}$.

e) If $c \neq 1$ and $x = x_0$, $y = y_0$ is a particular solution of the equation, then all solutions are given by $x = cx_0 - bk$, $y = cy_0 + ak$, where $k \in \mathbb{Z}$.

If $(a, b, c) = d$, then we divide equation by d and we repeat the previous procedure. It is important to note that we can write every particular solution in previous procedure by other adjacent fraction in Farey sequence.

That was the fourth method which we used in experimental study. It is important to emphasize that the students were acquainted only by first, fourth and fifth part of the algorithm. Here we will solve few examples of linear Diophantine equations with suggested algorithm.

Example 1. Find the solutions of $2x - 7y = 1$.

Here we have $a = 2, b = -7$ and $a < |b|$. From Farey sequence

$$F_7 = \frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}$$

we have $\frac{x_0}{y_0} = \frac{1}{4} < \frac{2}{7}$ and now the solution is given by $x = 4 + 7k$, $y = 1 + 2k$, where $k \in \mathbb{Z}$.

Example 2. Find the solutions of $7x + 2y = 3$.

Here we have $a = 7, b = 2$ and $a > b$. From Farey sequence

$$F_7 = \frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}$$

we have $\frac{x_0}{y_0} = \frac{1}{4} < \frac{2}{7}$ and now the solution is given by $x = -3 - 2k$, $y = 12 + 7k$, where $k \in \mathbb{Z}$.

Method – Study 1

Participants

All 124 seventh-grade students were from two state, urban primary schools and they voluntarily participated in this experimental study (61 from first school; 37 boys, 24 girls, 63 from second school; 30 boys, 33 girls). There were three seventh-grade mathematics classes at each school and 22 students were involved in advanced mathematics classes; 8 from first school, 14 from second school. Each school had a two seventh-grade mathematics teacher with 5 years of teaching experience from first and 6 years of teaching experience from second school and a master's degree in mathematics education. Students were divided in four groups; group A – 22 students from advanced classes, group B – 23 students, group C – 34 students and group D – 45 students. Students from the group A had prior knowledge in solving LDE with Euclid's algorithm, Euler method and guessing or inspection method, while the students from groups B, C and D had no prior knowledge in solving LDE.

Design and setting

The study was conducted with a mixed method research (concurrently); quantitative method with a pre – test for group A and post – test for all the groups, and qualitative method where we collected participants reflections and the main reasons for using the specific method to solve LDE. A mixed method research study uses both qualitative and quantitative methods either concurrently or sequentially (Venkatesh, Brown, and Bala, 2013). Pre – test for students in group A (see Figure 1 in Appendix) were given during the intervention.

Mathematics teachers whose students participated in the experimental study were acquainted by the properties of Farey sequence, mediant property and algorithm for solving the LDE during the professional development workshop, held in January 2018. Each group of students were acquainted with different numbers of methods for solving linear Diophantine equations (see Table 1). During the February 2018 students from both schools (groups B, C and D) were acquainted with Euclid's method, Euler method and guessing or inspection method by their teachers. During the March 2018 students from both schools (groups A and D) were acquainted with suggested method by their teachers. Data for the current study were collected in April 2018. Data was processed by IBM SPSS Statistics 20.

GROUP A	GROUP B	GROUP C	GROUP D
1. Algorithm for solving LDE with Farey sequences	1. Euclid's algorithm 2. Solving by guessing or inspection	1. Euclid's algorithm 2. Solving by guessing or inspection 3. Euler method	1. Euclid's algorithm 2. Solving by guessing or inspection 3. Euler method 4. Algorithm for solving LDE with Farey sequences

Table 1: Types of methods per groups

Data sources

As data source we used an individual pre – test and post – test. Post – test was design equally for each group of students (see Figure 2 in Appendix). Data collection from pre – test occurred in advanced classes over 30– min of classroom period. Data collection from post – test occurred in students mathematics classes over 60 – min of classroom period. At pre – test and post – test students were asked to solve each problem on their own. During the post – test students were concurrently conducted with qualitative research by questionnaire (see Figure 3 in Appendix). We wanted to get students reflections on the methodes they have been acquainted by and the main reasons for using specific method to solve LDE.

Results and discussion

Pre - test and post - test

Students from Group A had some algebra knowledge, recall that they learned in advanced classes Euclid's algorithm, Euler method and solving LDE with guessing or inspection. As shown in Table 2, students from Group A showed the best results in solving LDE with Euclid's algorithm (M = 2.82, SD = 0.733), and the worst results were by solving LDE with Euler method (M = 1.23, SD = 1.270). It is important to emphasize that there were eight students with zero points in second task, and all of them were from first school. Mathematics teacher from second school indicated that he usually encouraged use of multiple strategies in solving LDE at advanced classes and comparing solutions

		N	Mean	Std. Deviation
Euclid algorithm	First school	**8**	**2,88**	**,835**
	Second school	**14**	**2,79**	**,699**
	Total	**22**	**2,82**	**,733**
Euler method	First school	**8**	**,00**	**,000**
	Second school	**14**	**1,93**	**1,072**
	Total	**22**	**1,23**	**1,270**
Guessing or inspection	First school	**8**	**1,75**	**1,581**
	Second school	**14**	**1,79**	**1,051**
	Total	**22**	**1,77**	**1,232**

methods on regular mathematic class, but mathematics teacher from first school confirmed that he only uses Euclid's algorithm at advanced classes in solving LDE and that he doesn't have enough time on regular classes to use multiple strategies, to compare and discuss multiple solution methods. As shown in Table 3, there were no significant differences between first and second school in usage of Euclid's algorithm and solving with guessing or inspection, F (1, 20) = 0.072 and 0.950 respectively, but significant differences was found in using Euler's method between students from first and second school, F (1, 20) = 25.368.

Table 2: Students performance in pre -test

The first task in post – test enabled free choice for students to decide which method to use for solving a given linear Diophantine equation. Table 4 shows distribution of methods that students used to solve LDE in the first task in post – test. As shown, the suggested method in paper was most often used in the post – test. The main hypothesis for this research was that the groups that were acquainted with more methodes would get lower results in post – test, except the Group A. Ground for that hypothesis was the next research findings. Presenting too much material at once may confuse students because their working memory will be unable to process it. Working memory, the place where we process information, is small. It can only handle a few bits of information at once – too much information swamps our working memory (Rosenshine, 2012). Research by Akin (1998) showed that information overload may occur when too many topics are offered.

		Sum of Squares	df	Mean Square	F	Sig.
Euclid algorithm	Between Groups	**,041**	**1**	**,041**	**,072**	**,791**
	Within Groups	**11,232**	**20**	**,562**		
	Total	**11,273**	**21**			
Euler method	Between Groups	**18,935**	**1**	**18,935**	**25,368**	**,000**
	Within Groups	**14,929**	**20**	**,746**		
	Total	**33,864**	**21**			
Guessing or inspection	Between Groups	**,006**	**1**	**,006**	**,004**	**,950**
	Within Groups	**31,857**	**20**	**1,593**		
	Total	**31,864**	**21**			

Table 3: Differences between schools

Method/Group	GROUP A	GROUP B	GROUP C	GROUP D	TOTAL(%)
Euclid's algorithm	8	13	10	9	32
Euler method	0	-	8	2	10
Guessing or inspection	1	10	16	5	26
Solving with suggested algorithm	13	-	-	29	63

Table 4: Distribution of methods used to solve the first task in post – test per groups

The highest results on post – test was achieved by students in Group A, as predicted (M = 3.27, SD = 0.703). Results of Group C are lower than results of Group B, as predicted (M = 1.38, SD = 1.045, M = 2.22, SD = 1.126). Our hypothesis does not match up with the results from Group D (M = 2.91, SD =1.062). Recall, that students in Group D were acquainted by four methodes for solving linear Diophantine equations. This finding goes beyond the study of Lynch & Star (2014), where they found that six struggling students in the current study described quite positive experiences from learning algebra with a multiple-strategies approach. Students with average mathematics knowledge from Group D achieved result with no significant difference to the result of advanced students from Group A. As predicted, there was significant differences between groups $F(3, 120) = 20.409$, $p < 0.05$. Post hoc test (Gabriel) confirms the significant differences between Group A and Group B ($p = 0.004$), Group A and Group C ($p < 0.0005$), Group B and Group C ($p = 0.021$), Group B and Group D ($p = 0.044$), Group C and Group D ($p < 0.0005$). Significant differences weren't found between Group A and Group D ($p = 0.654$). So, the new question arises; *"What are the reasons that there is no significant difference between Group A and Group D?"* Results presented in Table 4 give us an indication that maybe the answer is hidden in usage of suggested method for solving LDE.

	Post - test		**Post – test without suggested method**	
Groups	Mean	Std. Deviation	Mean	Std. Deviation
Group A	**3,27**	**,703**	**3,27**	**,703**
Group B	**2,22**	**1,126**	**2,22**	**1,126**
Group C	**1,38**	**1,045**	**1,38**	**1,045**
Group D	**2,91**	**1,062**	**,71**	**1,236**

Table 5: Student performance in post – test

We will try to get an answer during the qualitative research which follows below. If we exclude the results derived from the use of suggested method in Group D, we get the results that corresponds to our hypothesis, $F(3, 120) = 28.717$, $p < 0.05$. Now, the post hoc test (Gabriel) confirms significant differences between all groups. Students that were exposed to higher number of methodes achieved lower results in Euclid's algorithm, Euler method and solving by guessing or inspection. It is interesting to point out that students from both Groups A and D achieved the highest results by using the suggested method, $M = 3.12$, $SD = 0.824$ and $M = 3.08$, $SD = 0.971$ (see Table 6). Students from second school achieved higher results in 2. a), 2. c) and 2. d) than the students from first school (see Table 6 and Table 7).

		a)		**b)**		**c)**		**d)**	
Groups	N	Mean	Std. Deviation	Mean	Std. Deviation	Mean	Std. Deviation	Mean	Std. Deviation
Group A	**8**	**2,88**	**,835**	**1,75**	**1,581**	**,00**	**,000**	**3,12**	**,824**
Group B	**11**	**2,07**	**1,003**	**1,92**	**1,409**				
Group C	**19**	**1,71**	**,867**	**1,87**	**1,521**	**1,21**	**1,111**		
Group D	**23**	**1.57**	**1,293**	**1,81**	**1,377**	**,87**	**,785**	**3,08**	**,971**
Total	**61**	**1,88**	**1,564**	**1,87**	**1,478**	**1,02**	**0,985**	**3,09**	**1,012**

Table 6: Students performance in second task (first school)

These findings confirms that using and comparison of multiple solutions help students move beyond rigid adherence to a single solution method to more adaptive and flexible use of multiple methods (Rittle-Johnson & Star, 2007). Students performance in third task confirms that the groups that were acquinted with more methods have achieved lower results. Students from both schools didn't succeed to solve task 3.d), see Table 8 and Table 9. This is the obvious proof that the suggested method has its own limitations. Students were unable to determine the Farey sequence F_{2018} (see Figure 4). Ivan and Dinko wrote below the F_{2018}; “it was too little time to find the adjacent fraction to $\frac{11}{2018}$. It is impossible to solve this task”. When students finished a post – test, a web link to Farey fractions calculator was given in order to help them to solve LDE that was given in third task.

		a)		b)		c)		d)	
Groups	N	Mean	Std. Deviation	Mean	Std. Deviation	Mean	Std. Deviation	Mean	Std. Deviation
Group A	**14**	**2,79**	**,799**	**1,79**	**1,051**	**1,93**	**1,072**	**3,58**	**,414**
Group B	**12**	**2,24**	**1,107**	**1,87**	**1,369**				
Group C	**15**	**1,69**	**,743**	**1,76**	**1,489**	**1,41**	**1,302**		
Group D	**22**	**1,54**	**1,234**	**1,75**	**1,141**	**1,45**	**1,255**	**3,14**	**1,071**
Total	**63**	**2,12**	**1,544**	**1,84**	**1,596**	**1,57**	**1,575**	**3,31**	**1,115**

Table 7: Students performance in second task (second school)

		a)		b)		c)		d)	
Groups	N	Mean	Std. Deviation	Mean	Std. Deviation	Mean	Std. Deviation	Mean	Std. Deviation
Group A	**8**	**2,82**	**,885**	**1,49**	**1,331**	**,00**	**,000**	**,00**	**,000**
Group B	**11**	**1,87**	**1,122**	**1,12**	**1,247**				
Group C	**19**	**1,47**	**,955**	**0,84**	**1,202**	**1,19**	**,853**		
Group D	**23**	**1,40**	**1,271**	**0,75**	**0,981**	**,47**	**,423**	**,00**	**,000**
Total	**61**	**1,69**	**1,349**	**0.94**	**1,478**	**,80**	**0,584**	**,00**	**,000**

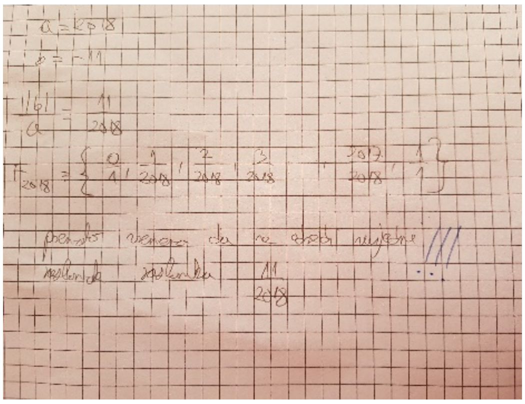
Table 8: Students performance in third task (first school)

		a)		b)		c)		d)	
Groups	N	Mean	Std. Deviation	Mean	Std. Deviation	Mean	Std. Deviation	Mean	Std. Deviation
Group A	**14**	**2,87**	**,846**	**1,64**	**1,315**	**1,53**	**1,001**	**,00**	**,000**
Group B	**12**	**2,01**	**1,011**	**1,17**	**1,403**				
Group C	**15**	**1,41**	**,706**	**,80**	**1,286**	**1,37**	**1,262**		
Group D	**22**	**1,63**	**1,274**	**,81**	**1,177**	**1,31**	**1,227**	**,00**	**,000**
Total	**63**	**2,08**	**1,349**	**1,10**	**1,621**	**1,03**	**1,349**	**,00**	**,000**

Table 9: Students performance in third task (second school)

Questionnaire

Distribution of student’s answers from the first question in the questionnaire is given in the Figure 5. Figure 5 corresponds to the results that was given in Table 6 and Table 7, as also in Table 8 and Table 9. Students from second school achieved better results in the post – test because they have been often exposed to multiple strategies on regular mathematics classes. There is a lack of study's that assessed the influence of the multiple strategies on student learning gains in mathematics in a way to link this teaching practice to measurement of student outcomes, especially in the domain of solving linear Diophantine equations. Answer to second question from questionnaire was given partially in Table 4. Main reasons for using arbitrary method or their advantages listed in Table 10 corresponds to exploratory study conducted by Lynch & Star (2014). Now we can answer to a question that arises



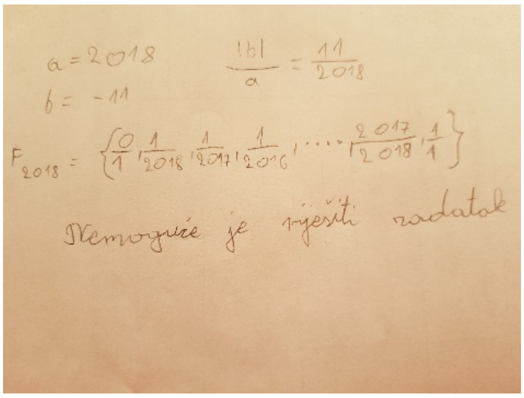


Figure 4: Ivan's and Dinko's solution's

from Table 5; *"What are the reasons that there is no significant difference between Group A and Group D?"* If we extract the answers from Group D for solving with suggested algorithm, we get

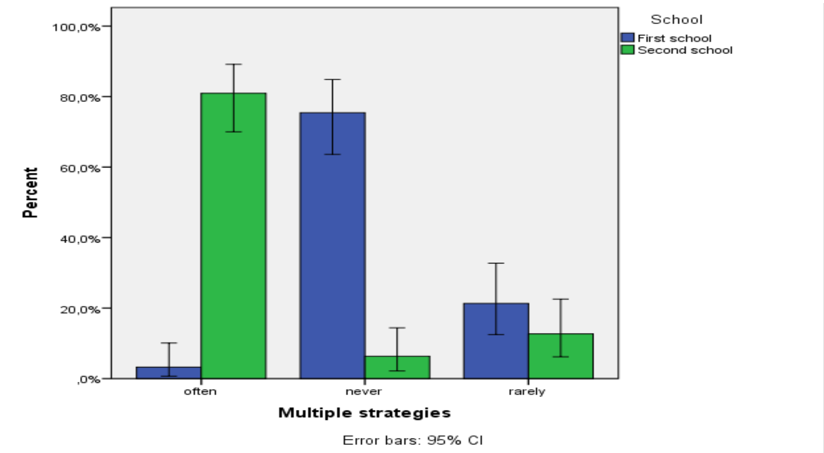


Figure 5: Distribution of the students answer from the first question in the questionnaire

following list: "I only get correct solution using this method" (N = 11), "This method works for me" (N = 7), "Allows me to solve equation more efficiently" (N = 8), "This was only method I wasn't frustrated by" (N = 11). From that we conclude that teaching with multiple strategies helped low and average achieving students to find the "best method for them" or "way that works for them". Table 11 shows the explanations for the fifth question from the questionnaire. As predicted, students had problem in third task to find Farey sequence F_{2018}. Students from Group A and D failed to get at least one point in task 3.d), so we can strongly confirm the limitations of suggested method. All of disadvantages for suggested method are from category of mathematical understanding. Fourteen students from Group A and Group D confirmed that they just need one method that works for them. As shown in Table 11, students mostly list their feelings as disadvantages. We can sort their feelings/disadvantages in four categories: 1) risk of confusion - more than fifty percent of students (52%), 2) students' resistance to learning multiple strategies – more than fifty percent of students (53%), 3) affective and motivational issues – closely linked to resistance, affective and motivational issues (55%) and 4) mathematical understanding. This finding corresponds to research of (Akin, 1998; Lynch & Star, 2014).

Method	Advantages
Solving with suggested algorithm	• "This method works for me" (N = 7) • "I only get correct solution using this method" (N = 11) • "This was something different in a positive way" (N = 8) • "Allows me to solve equation more efficiently" (N = 8) • "This method reduced boredom of previous methods" (N = 8) • "This was only method I wasn't frustrated by" (N = 11)
Euclid's algorithm	• "This method seems to be reliable" (N = 16) • "Simple method that has clear way from beginning to the end" (N = 20) • "If I divide two numbers, half of job is finished" (N = 10)
Solving by guessing or inspection	• "If you guess the particular solution, after that the rest of solution is obvious" (N = 23) • "Suprisingly the simplest method" (N = 14)
Euler's method	• "This method seems to be reliable" (N =7) • "This method is effective" (N =6)

Table 10: Advantages of methodes from student's point of view

Method	Disadvantages
Solving with suggested algorithm	• "For a large a and b it is impossible to find Farey sequence without a help of computer" (N = 67) • "When I find adjacent fraction, I don't know how to write the solution" (N =19) • "Euclid's algorithm is enough for me" (N = 14)
Euclid's algorithm	• "You need to much time to find particular solution" (N = 13) • "I was confused by others methods" (N = 47) • "I mixed up methods" (N = 28)
Solving by guessing or inspection	• "Unreliable" (N = 39) • "I was confused by others methods" (N = 41) • "I mixed up methods" (N = 2) • "Euclid's algorithm is enough for me" (N =79) • "I give up, after I understand first method" (N = 42)
Euler's method	• "I was confused and frustrated by others methods" (N = 64) • "I mixed up methods" (N =67) • "I give up, after I understand first method" (N = 83)

	• "Euclid's algorithm is enough for me" (N = 75)

Table 11: Disadvantages of methodes from student's point of view

Method – Study 2

After the Study 1 finished, mathematics teachers from each school held two additional lessons at the end of the school year for Group B and Group C in order to show them the rest of methods, Euler method and suggested method. One year later we obtained research in order to measure a retained knowledge for methodes used in Study 1. Knowledge retention involves capturing knowledge in the organizational memory so that it can be used later as presented in research of Walsh and Ungson (1991). Also, the procedural flexibility was assessed. Procedural flexibility incorporates knowledge of multiple ways to solve problems and when to use them (Star, 2005).

Participants

Study was conducted just on first school, where there is no difference in group of 61 students from Study 1. Now we have students who have prior – knowledge in all four methods for solving LDE.

Design, setting and data source

For measuring the retained knowledge and procedural flexibility a post – test is constructed (see Table 12). Similar test for procedural flexibility was designed in research by Rittle – Johnson & Star (2007). Data for current study was collected in March 2019.

Generating multiple methodes (GMM)	1. Solve this equation in two different ways: $5x - 3y = 7.$	2 pts for each correct solution
Recognize multiple methodes (RMM)	2. Linear Diophantine equation is given $4x + 3y = 5.$ Identify all possible steps that could be done next: a) 4 : 3 = 1 1 b) $y = \frac{5-4x}{3}$ c) $x = 2, y = -1$	4 pts, 1pt for each step 3 pts, 1pt for each step 2 pts, 1pt for each step
Evaluate suggested method (ESM)	3. $F_4 = \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}$ $\frac{x_0}{y_0} = \frac{2}{3}$ a) What steps did the student used in first and second line? b) Do you think that this way of starting this problem is: A. a very good way	1pt if correctly identify each step

	B. ok to do, but not very good way C. not ok to do	1 pt for choice A

Table 12: Sample items for assessing procedural flexibility and retained knowledge

Results and discussion

In assessment procedural flexibility and retained knowledge, 10 students decided to solve first task by Euclid's algorithm and solving by guessing or inspection (M = 1.21, SD = 0.69), 28 with Euclid's algorithm and suggested method (M = 1.06, SD = 0.34) and 23 with suggested method and solving by guessing or inspection (M = 1.45, SD = 0.57). None of the students has chosen a Euler's method. As shown in Table 13, suggested method was the most successful method. Almost half of the students

Method/Item	First item	Second item, a)	Second item, b)	Second item, c)	Third item, a)	Third item, b)
Euclid's algorithm	46/76	119/244	-	-	-	-
Euler method	-	-	12/183	-	-	-
Guessing or inspection	35/66	-	-	88/122	-	-
Solving with suggested algorithm	73/102	-	-	-	107/122	60/61

Table 13: Students performance in post – test

that used suggested method got two points (47%), only 9% that solved LDE by guessing or inspection got two points, and 37% of students got two points using Euclid's algorithm. All columns show total points achieved with stated methods. Twenty six percent of students didn't recognize that 2. a) is a beginning of Euclid's algorithm, but 30% of students found all four steps. Thirty one percent of students didn't recognize that 2. b) is a beginning of Euler's method, but the rest of them (69%) found the next step correctly. Twenty five percent of students didn't recognize that 2. c) is a beginning of guessing or inspection method, but 57% found next two steps correctly. Seventy seven percent of students found correctly next two steps of suggested method in item 3. a) and only one student didn't succeed to identify none of the two steps. In item 3. b) only one student didn't choose correct answer. *Retention of knowledge.* Students who used the suggested method achieved highest score in first item (72% of points), and the lowest score was achieved by using guessing or inspection method. That implies, suggested method provides the greatest knowledge retention (see Table 13).

Procedural flexibility. Evaluation of the suggested method is a flexibility component with the highest final score (see Figure 6). This finding is similar to the research of Rittle – Johnson & Star (2007), where they got the highest score in generate component and the lowest score in recognize component.

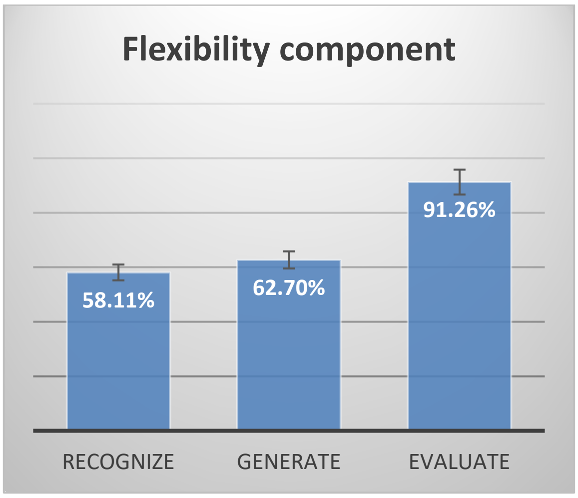


Figure 6: Assessment for three components of procedural flexibility

Conclusion

Goal of this research was to get the answer on the following questions. *Can using multiple strategies lead to greater gain in solving LDE, or does it lead to confusion? What are the main reasons for using the specific method to solve LDE from students' point of view?* These research shows that students who are exposed to multiple strategies for solving LDE may become confused and may developed resistance to multiple strategies but at the same time it helps to ensure that each student found "a method that works for him". Without careful consideration of how to teach with multiple strategies, including keen attention to the choice and presentation of methods, students may indeed become confused. This research suggests, that with these elements, low and average achieving students may benefit from this instructional approach. Study 2 showed high level of retention knowledge with suggested method and high level of procedural flexibility components. General, limited research has been done on what teaching with multiple strategies should look like, so that this study provides empirical evidence that in learning to solve linear Diophantine equations, it pays to be exposed to multiple strategies. In the hope to improve implementations of multiple strategies in mathematical curriculum, I hope that these findings can induce future research on applications of multiple strategies in mathematics.

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Appendix

Find the solutions of linear Diophantine equation $3x - 10y = 2$, by a) using the Euclid's algorithm, (4 pts) b) using the Euler method, (4 pts) c) guessing or inspection. (4 pts)

Figure 1: Pre – test

1. Find the all solutions of linear Diophantine equation $2x + 7y = 4$ using the arbitrary method.	(4 pts)
2. Find the all solutions of linear Diophantine equation $8x - 7y = 4$ using the	
a) Euclid's algorithm,	(4 pts)
b) guessing or inspection,	(4 pts)
c) Euler method (only for A, C and D group),	(4 pts)
d) algorithm for solving LDE with properties of Farey sequence (only for A and D group).	(4 pts)
3. Find the all solutions of linear Diophantine equation $2018x + 11y = 4$ using the	
a) Euclid's algorithm,	(4 pts)
b) guessing or inspection,	(4 pts)
c) Euler method (only for A, C and D group),	(4 pts)
d) algorithm for solving LDE with properties of Farey sequence (only for A and D group).	(4 pts)

Figure 2: Post – test design

Underline the school you are from: FIRST SCHOOL SECOND SCHOOL

1. In your regular mathematics classes, were you frequently exposed to multiple strategies for solving math problems?
   a) often b) never c) rarely
2. What method did you used to solve linear Diophantine equation in first task? Can you list the advantages of that method?
3. What methods you didn't know to apply in second task?
4. What methods you didn't know to apply in third task?
5. Can you explain your answer in two previous questions in a way to list the all disadvantages you encounter to?

Figure 3: Questionnaire

The application of modern technology in teaching and learning stereometry

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The paper analyzes the possibilities of applying a three-dimensional view of geometric objects within the dynamic software GeoGebra, during the teaching and learning stereometry. The research included students in the final (eighth) grade of elementary school. The GeoGebra software was applied in introducing students in the formation and the elements of a right circular cylinder. The method of collaborative learning was applied. The students worked in small groups and they had the possibility to independently create dynamic worksheets, by using GeoGebra software, which they used for observing a three-dimensional view of solid figures (cylinder and prism), analyzing and comparing their elementary properties. After finishing group work, the students presented their results, after which they had the opportunity to discuss their observations. The results of students' work were analyzed. Guidelines for further research as well as for the implementation of the described approach in teaching are given.

Keywords: Cylinder; Dynamic software; Stereometry; Three-dimensional view.

Introduction

Solid geometry (stereometry) is a very important mathematics topic in the most of the curricula. Its studying usually starts during elementary education and it is also present in secondary and tertiary education. Within this topic, the students are being introduced to the solid figures (cube, cuboid, prism, pyramid, cylinder, cone, sphere...) and their properties.

In Serbian educational system, stereometry contents are being studied during primary education, when the most of students are about 10 years old. At this level, students become familiar with the cube and cuboid, their surface areas and volumes. Also, the students are being introduced to the concepts of cylinder, cone and sphere. It should be mentioned that children are being introduced to the solid figures and their names earlier, even during preschool education. Later, at final grade of elementary school (when the most of the students are 14 years old), solid figures and their properties are being studied more in detail. At that level, the students are being introduced to right figures (actually, they learn what oblique figures are, but not in detail). Finally, in the third grade of high school, they are being introduced to solid figures and their properties much in detail.

The teachers and the students usually encounter difficulties in learning solid geometry. The students have problems with understanding the contents and the teachers have problems to explain them different concepts and their properties in this area. Most of these difficulties are related to the spatial reasoning. Usually, students have problems to identify all necessary elements of the observed solid figures, to understand their generation, to perceive a net of individual figure. Especially, the students have difficulties to perceive an intersection of the figure and plane. Earlier research and experience showed that some of these difficulties are usual for many students, even for those who master the material at the advanced level of achievements. It can be said that the students have problem to imagine the figures they don't see.

Usually, the teachers try to find different ways to overcome these difficulties. One of the most important things is visualization. The teachers have always been applied different methods to visualize solid figures. The sketch is usual, but not so effective, because it is not clear enough when we show 3D object in two dimensions. Better results have been achieved when three-dimensional models or wear-frame models were used. Especially wear-frame models, because their application enables different elements, such as diagonals and intersections, to be shown. However, the application of modern technology creates the new possibilities in visualization.

In this paper, some possibilities of using modern technology in teaching and learning stereometry are presented, and the influence of its application on students' understanding of a cylinder and its properties is analyzed.

Theoretical background

The understanding of the solid figures and their properties depends on the spatial reasoning and spatial ability of the individual. Students' spatial abilities, consisted of spatial visualization, orientation and relations factors, are a strong predictive factor of the reasoning in solid geometry thinking (Pittalis & Christou, 2010). Clements and Battista (1992) claim that "spatial reasoning consists of the set of cognitive processes by which mental representations of spatial objects, relationships, and transformations are constructed and manipulated".

Earlier studies have shown that spatial reasoning intervention improves the students' performance in geometry, especially solid geometry (Pittalis & Christou, 2010; Ramful, Lowrie, & Logan, 2017). It is shown that improvement in children's spatial ability has a positive influence to their mathematics ability in general (Cheng & Mix, 2014).

However, numerous difficulties are usually encountered in teaching and learning stereometry contents. Many students have difficulties to understand and describe the properties of 3D objects, draw the correct two-dimensional representation and the net of the observed object (Marchis, 2012). Also, many students, especially those who mater the material on the elementary or intermediate level of achievement, have problems to understand the generation of the solid figures and to perceive an intersection of the figure and plane. This is also difficult topic for teachers, because they have to help their students in overcoming encountered difficulties (Salman, 2009).

Most of the teachers try to find a way to explain difficult teaching contents and to provide a suitable learning environment to their students. Usually, the teaching of solid geometry is being enhanced by improving the multiple representations of the observed objects. When it comes to the stereometry, the most importance is given to the representations which enable visualization of the solid figures because it is considered that visualization has a significant importance in describing and explaining the properties of solid figures (Kurtuluş, & Uygan, 2010). Particular importance is given to the realistic representations, such as 3D models of the solid figures and, especially in recent times, to the manipulative representations (Nakahara, 2008).

Multiple representations in teaching mathematics have gained significance with the development of technology. Improving the teaching of mathematics and, especially, stereometry, by using appropriate software, was the topic of many researches (Baki, Kosa, & Guven, 2011; Hwang & Hu, 2013; Stols, 2012). Most of the software packages, which are appropriate for using in teaching and learning stereometry, enable multiple representations of solid figures, with the accent on graphical representation. Visualization of solid figures is being further enhanced by the possibility of rotating the observed figure. Also, dynamic geometry software enables creating the manipulative representation of the observed object. Manipulative representation provides a possibility that measures of some elements of the observed solid figure be defined by slider, which moving causes the changes in the measure of depended element. As a result, some properties of the observed figure are being changed. The use of manipulative representation enables analysis of the influence of individual elements to the properties of the observed solid figure.

The application of the physical manipulatives and dynamic geometry software in teaching is more effective in developing the students' spatial visualization skills than traditional instruction. Also, instructions based on dynamic geometry software proved to be more effective than the instructions based on the physical manipulatives (Baki, Kosa, & Guven, 2011). The use of dynamic software, adjusted for dealing with solid figures, stimulates geometric cognitive growth of the pre-service teachers. Also, it can contribute to better understanding the properties of solid figures, especially when it comes to students who master the material on elementary and intermediate, or upper-intermediate level of achievements (Stols, 2012).

The use of software in a collaborative environment contributes to the students' ability in geometric problem solving, especially when it comes to the 3D objects. Namely, software which provides multiple representations of the observed object contributes enable properties of that object to be observed from different points of view and peer learning, which is present in a collaborative learning environment, is useful to facilitate geometric problem solving by sharing ideas and exploring multiple representations (Hwang & Hu, 2013).

One of the dynamic software packages, which are appropriate for creating the multiple representations of the solid figures, is *GeoGebra*. It enables 3D view, including manipulative representation of the object created in this view. This educational software proved to be successful in teaching geometry, but also other branches of mathematics. The use of *GeoGebra* software for visualization and working within multiple representations environment was the topic of the numerous studies (Arzarello, Ferrara, & Robutti, 2012; Božić, Takači, & Stankov, 2019; Lazarov, 2012; Tran, Nguyen, Bui, & Phan, 2014), which proved its benefits.

The application of *GeoGebra* software contributes to the successful implementation of problem-based and discovery learning. It helps the teachers in realization the teaching process, but also helps the students during individual learning and improves the interaction between teacher and students (Tran et al., 2014). Dynamic representations in *GeoGebra* environment can be successfully applied in mathematical modeling in teaching, i.e. in problem solving activities, for creating approximations of the models (Arzarello, Ferrara, & Robutti, 2012). It is proved that the application of *GeoGebra* and the use of dynamic multiple representations in *GeoGebra* environment significantly contributes to the students' better achievements in examining functions with parameters (Božić, Takači, & Stankov, 2019).

Besides *GeoGebra*, there are some other software packages appropriate for the use in teaching stereometry. Opposite most of them, *GeoGebra* is free software, so the students and the teachers don't have difficulties with its availability. Also, *GeoGebra* is easy to use and the students can be easily trained for its application (Hohenwarter, Jarvis, & Lavicza, 2009).

Research question

Previous research showed that the application of dynamic geometry software with possibility of 3D view can improve students' spatial ability (Baki, Kosa, & Guven, 2011; Stols, 2012). Also, it is proved that work in a collaborative environment has a positive influence on students' achievements in learning geometry (Hwang & Hu, 2013). Because of all mentioned, it is decided to examine the influence of the application of dynamic software *GeoGebra* within a collaborative learning environment on students' understanding of the cylinder concept and its properties.

The main question of this research is:

Does the application of dynamic geometry software with 3D view, within collaborative learning environment, contribute to students' better understanding the cylinder concept, its generation, elements and properties?

Methodology

This research presents a case study. The research is conducted with 24 students of the final (8^{th}) grade of the Elementary school "Vuk Karadžić" in Novi Sad, Serbia, in the 2017/2018 school year. At first, the students were divided into four-member collaborative groups. Eight collaborative groups were formed at the beginning of the class, taking into account students' previous achievements so in each group there were students who reached elementary, intermediate and advanced level of achievements and the heterogeneity of the groups has been achieved. They worked in the digital classroom and every group used one computer.

The study covered an introduction with the cylinder concept and its properties and students worked two school hours (90 minutes). A cylinder concept and its properties were chosen as a teaching topic to be processed within this experiment. *GeoGebra* software package was chosen for creating the dynamic multiple representations of the observed solid figure. Qualitative analysis of the students' work is carried out. The students work within *GeoGebra* environment was observed by the teacher. Cooperation and discussion among the students in every collaborative group, but also discussion among the groups and with teacher was the subject of observation. The dynamic worksheets, created by the students within *GeoGebra* environment, were saved and analyzed by the teacher.

The analysis of the students' work

After forming the collaborative groups, the students were introduced with the topic. The concept of a cylinder is not unknown to students of the final grade of elementary school (they are able to recognize solid figures during primary education). So, at first, previously learned was recalled and students discussed about cylinder and its properties with the teacher and with each other. Most of the students tried to define a cylinder by explaining its properties. After that, they were introduced in the cylinder concept in detail; with the accent on the right cylinder (oblique cylinder was only mentioned – it is being studied in detail in high school). In this part of the lesson, the dialog method was dominated. But, this method was present during all 90 minutes, less or more.

After the introduction, the teacher provided necessary instructions for students. They are informed about what they are expected to do and how to use *GeoGebra* software. The eighth grade students are usually familiar with *GeoGebra*, because they learn about this software within computer science. The teacher gave the instructions about the use of 3D view in *GeoGebra* and suggested the order by which students should analyze cylinder and its elements.

In the most of the groups, the students organized their work in such way that one of them worked within *GeoGebra* dynamic worksheets, the other gave the instructions and the rest two noted the results. During the work process, the students usually changed their roles within the groups, so every student had the opportunity to work with *GeoGebra* software. In all the groups, at the beginning, the student with higher achievements was in charge for giving instructions, but later, during the learning process, the other students were giving their suggestions and they made the conclusions together, by discussion.

In the next phase, the students discussed about the generation of a cylinder. The teacher had to show them generation of a cylindrical surface, but some of the students knew that cylinder can be generated by rotation of a rectangle. They explained that a cylinder can be generated by "rotation of a rectangle around its side." After the teacher's instructions, the students used *GeoGebra* software in order to create dynamic worksheets which show generation of a cylinder by rotation of a rectangle. When they had some difficulties, they could get help of the teacher. One example of these worksheets, created by the students, is shown in Figure 1. After they used *GeoGebra* dynamic worksheets to observe generation of a cylinder, some students noticed that the cylinder also can be generated by rotation of a rectangle around its symmetry axes, but also around any line which is parallel to its side.

After that, the students observed different intersections of a cylinder and a plane and discussed about the figures obtained. In this part, the students also used *GeoGebra* software performances in the best possible way. They used the appropriate options of *GeoGebra* 3D view and created the representations of the requested intersections. They created different intersections and discussed the about which figures could be the intersections of a cylinder and a plane. Most of the students observed the cases when the intersection is a rectangle or a circle, especially the case when the observed plane contains the axis of the cylinder. One example of such students' work is shown in Figure 3.

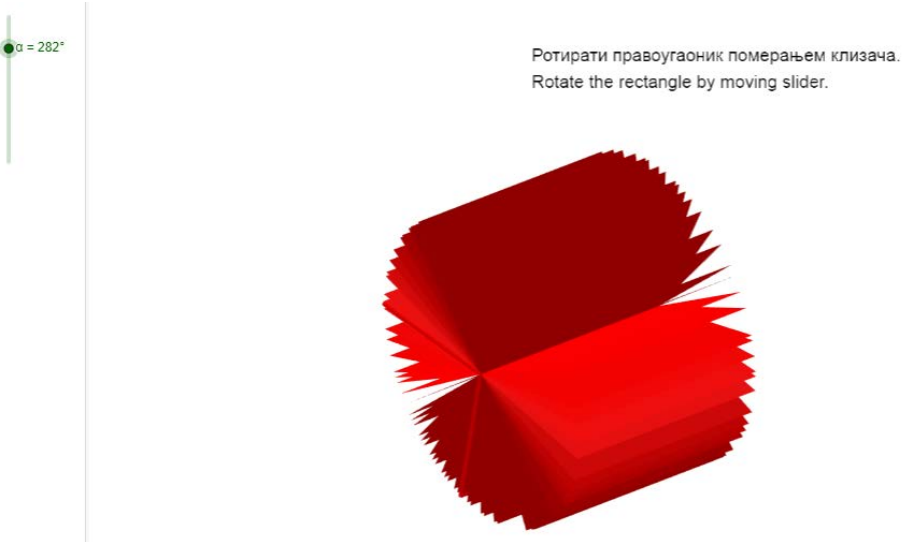

Figure 1: Students' work – rotation of a rectangle

They concluded that there are infinitely many possibilities of the intersection with a plane, but, if the plane contains an axis of a cylinder, all those intersections (rectangles) are congruent and in the other cases they are non-congruent. The students who observed the intersections of a cylinder with a plane parallel to the planes of the bases concluded that those intersections are the circles. Three groups of students observed the other cases and concluded that, in those cases, intersections of a cylinder with a plane can be some "irregular" figures (they used term irregular because those figures were unknown to them). Only one student recognized the ellipse as the intersection.

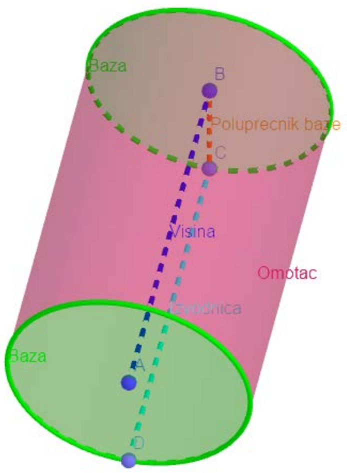

Figure 2: Students' work – elements of a cylinder

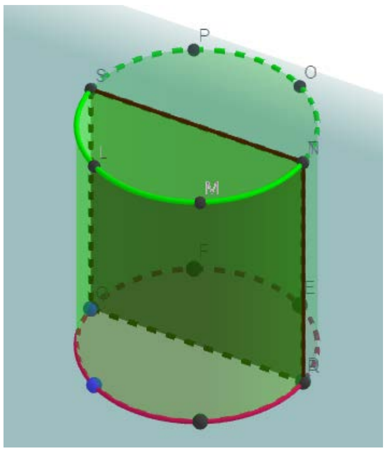

Figure 3: Students' work – intersection with a plane

The next task for the students was to analyze a net of a cylinder. Some of the students had an idea how a net looks like, but they couldn't create it within *GeoGebra* software. This was expected, because it is too complicated for them. Namely, for creating a net of a cylinder, knowledge of elementary school students is not enough. Because of that, the teacher showed them a net of a cylinder, created within *GeoGebra* environment and they discussed about it. As a result of this discussion, the students concluded how to determine the surface area of a cylinder. They explained that the surface area of a cylinder presents a sum of the surface areas of its bases (two circles) and the surface area of its lateral area (rectangle, if a cylinder is "opened").

By following teachers' instructions, the students created new worksheets, which are used for the comparison of a cylinder and a prism. The students were familiar with the prism and its properties, and now they compared it with the cylinder properties. In Figure 4, an example of comparison between cylinder and prism is shown. Many students noticed that "when the number of prism's sides is being increased, a prism looks more and more like a cylinder". One student said that "cylinder can be considered as a prism with the infinitely many sides." As a result, they concluded how to determine the volume of a cylinder, i.e. they explained that the volume of a cylinder can be determined in the same way as the volume of a prism.

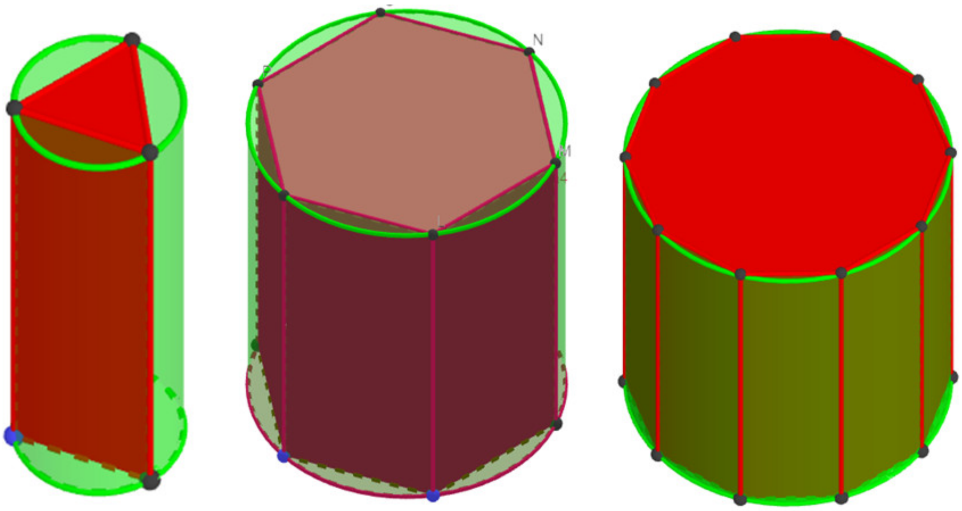

Figure 4: Students' work – cylinder and prism

During the learning process, there was intense discussion among the students and, also, they asked a number of questions to the teacher. All students actively participated in the learning process and discussion, which is not the case when a classical teaching method is applied. Usually, when the classical teaching method is applied, only several students participate in discussion and the others passively follow the lesson. Also, there were no students who were not motivated to follow the lesson. The students also were interested in *GeoGebra* 3D view and its possibilities and the teacher informed them about appropriate information source and the literature. The analysis of their *GeoGebra* dynamic worksheets showed that they have an excellent deal with this software package.

Conclusions

During two classes which were covered by the research, the students work was carefully monitored by the teacher. Special attention is paid to the students' work within collaborative groups, the use of *GeoGebra* software, but also their mutual cooperation, discussion and exchanging ideas and the organization of the group work, in general. Besides that, discussion and cooperation between different groups were also observed and analyzed.

At the beginning of their work within *GeoGebra* environment, the students split their roles it the way described above. Later, during the work, they changed their roles and adjusted them to the current needs. All students, in all the groups, were included in the group work and all of them performed their duties appropriately. It is important to mention that the students with lower achievements, at the beginning, were unsure of themselves, but later they also participated in the group work, not less than the other group members. These students usually worked within *GeoGebra* dynamic worksheets, following instructions from the other students. During the work, they were discussing in order to reach the correct conclusions.

At the beginning, when they some difficulties, especially when it comes to the use of *GeoGebra* software, the students of some group asked the teacher for help. Later, they usually asked the members of the other groups. In this way, cooperation between different groups has been achieved. Also, during the discussion about the results of their work, the students of different groups expressed their opinions and disagreements with the conclusions of the other students.

As regards their conclusions about the properties of a cylinder, its generation, elements and intersections, the students reached all necessary conclusions, more independently and with significantly less help of the teacher than it is the case when classical teaching methods are applied. Besides, some of the students came to the ideas and reached the conclusions which are usually not expected from the students of their age.

From the above, it can be concluded that the use of software helped students to visualize the observed solid figure and to perceive its properties from different points of view, by using dynamic multiple representations. Also, the collaborative learning method, applied in the observed learning process, helped students to correct some mistakes and to, through the exchange of their ideas, reach the correct conclusions. Finally, it can be concluded that the application of dynamic geometry software with 3D view, within a collaborative learning environment, contributes to students' better understanding the cylinder concept, its generation, elements and properties, which presents positive answer to the research question. Dynamic geometry software, which enables 3D view, should be applied in teaching solid geometry contents, when there are conditions for its application and when a teacher estimates that it could be helpful. Also, collaborative learning method can be useful in teaching and learning these contents, because it enables discussion and exchanging the ideas among the students, which encourage active participation of every student in the learning process.

Some future research should be conducted with a larger sample, with the experimental and the control group. The participants of the research should be tested and their test results should be quantitatively analyzed. The data obtained in that way would confirm if the influence of the application of dynamic geometry software causes statistically significant advantage in the achievements of the students who used the computer in comparison with the students who learned in a classical way. Also, the high school students should be covered by the future research, because of the difficulties occurred in teaching and learning stereometry contents in the high school, but also it should be examined the influence of modern technology application on the achievements of the younger students, at the beginning of their introduction with the stereometry concepts.

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Pre-service primary education teachers' knowledge of relationships among quadrilaterals

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Students obtaining a teaching degree in primary education should themselves acquire a certain level of knowledge of mathematics and develop their own mathematical competencies, in order to ensure appropriate learning environment for development of primary school students' mathematical knowledge and competencies. Various educational research reveal that many university students encounter difficulties with tasks involving geometrical concepts, often misunderstanding the concepts, especially the relationships among the various quadrilaterals. Special attention is devoted to understanding the hierarchical classification of quadrilaterals. The research presented in this paper was conducted with third year undergraduate students of Primary Education at the University of Split Faculty of Humanities and Social Sciences. The aim of the research was to establish the scope of primary education students' knowledge of quadrilaterals, the manner in which they establish relationships among quadrilaterals as well as to investigate students' misconceptions of these relationships. Obtained results are comparable to results of other researchers and confirm difficulties in identifying quadrilaterals in non-standard positions or of non-standard shapes, in addition to unexpectedly poor knowledge of quadrilaterals' properties and inability to establish inclusion relationships, especially in the case of trapezium.

Keywords*: Geometry thinking, Hierarchical classification, Inclusion definition, Primary education, Quadrilaterals.*

1. Introduction

Children take their first steps in mathematics during primary education, and in order to complete these steps they need to be supported by their teachers. This period of child development is rather delicate, as children encounter mathematical ideas and concepts and mathematical problem-solving for the first time at the beginning of their education, starting to develop mathematical and logical reasoning (Žilková, 2015). Children develop emotional relationships with the teachers and grow to love the subject through the teacher; therefore, the role of the teacher is far more important than in later stages when children reach independence. Thus, primary education teachers have great responsibility to provide sound foundation for continuation of education.

In order to succeed in this endeavour, teachers should have thorough knowledge and understanding of subject matter of mathematics, as well as wide range of teaching methods, to be able to bring the subject matter closer to the learners in the best possible way. Therefore, students preparing for work in primary education receive training consisting of two important parts. The first part is focused on subject matter knowledge and the second part on pedagogical aspects of teaching. The basic precondition for successful adoption of different strategies and teaching methods in mathematics is content knowledge and understanding of mathematical ideas, concepts and their relationships (Gűnhan, 2014). One important part of mathematical education is geometry, geometrical concepts and their relations, as learning and teaching geometry contributes to the development of visualisation skills, spatial thinking, deductive reasoning, argumentation and proving (Fujita & Jones, 2007).

Various educational research show that many students experience difficulties specifically when working with geometrical concepts, misunderstanding these particular concepts. Special attention is devoted to difficulties in establishing relationships between quadrilaterals and understanding the hierarchical classification of quadrilaterals (e.g. De Villiers 2009, Fujita & Jones, 2007; Gűnhan, 2014, Žilková, 2015). Furthermore, limited understanding or misunderstanding of the concepts and their relationships is difficult to correct, and in some cases “resistant to change” (De Villiers, 2010, p. 572).

In order to influence the development of geometrical reasoning in their students, teachers should first develop their own geometrical reasoning, followed by acquiring knowledge of strategies for developing geometrical reasoning and factors which foster or hinder such development. Quadrilaterals are an ideal subject matter for developing geometrical reasoning, appropriate for developing the skills of identifying geometric shapes in different forms and positions, analysing their properties and establishing appropriate relationships based on these properties. According to van Hiele, this is exactly the process required for development of geometrical reasoning (van Hiele, 1986).

Quadrilaterals are first introduced to students in the Croatian educational system and the approved textbooks in the 4th grade of primary school, starting with squares and rectangles, continuing in the 6th grade with general knowledge on quadrilaterals, concept of parallelogram, rhombus, rectangle and square as special types of parallelogram. Afterwards, the curriculum covers trapezium and its relationship with parallelograms. Deltoid is mentioned only in relation to the concept of area of quadrilateral, without further study of its relation to other quadrilaterals. In other grades of primary school and 1st and 2nd grade of secondary school, knowledge of quadrilaterals is revised and broadened through various topics in plane geometry, with emphasis on tasks with different levels of complexity.

The aim of this paper is to present results of research conducted with third year undergraduate students of primary education, concerning their knowledge of quadrilaterals: can they identify specific types of quadrilaterals, which properties are they familiar with and how do they establish relations between them. Theoretical background for assessing the achieved levels of reasoning in students is the van Hiele theory on the development of geometrical thinking in five levels. As there is no previous research conducted in Croatia with this focus, the results provide insight into students’ knowledge of quadrilaterals acquired after 12 years of math education. Based on the results of this research, in spite of the small sample size, guidelines and recommendations for further work can be given.

2. Theoretical background

According to the results of other research, many primary and secondary level students, as well as university students encounter difficulties in understanding geometrical concepts, especially in forming inclusion relations between concepts. Many of the existing misconceptions prevent the development of geometrical reasoning, hence some researchers give guidelines for overcoming these difficulties (e.g. Abdullah & Zakaria, 2013; Armah, Cofie and Okpoti, 2018, De Villiers, 2009; Fujita and Jones, 2007; Erez & Yerushalmy, 2006; Graumann, 2005; Gűnhan, 2014; van Hiele, 1999).

In order to interpret the results of the research and the resulting guidelines, it is important to explain the meaning of terms used in this paper, which are necessary for investigating the students' knowledge of quadrilaterals.

The van Hiele theory

The van Hiele theory is constructed from three parts: (a) description of development of abstract reasoning, hierarchically through five levels; (b) discussion on characteristics of the five-level model; (c) description of the learning process in five phases, allowing advancement from one level of thought to another (van Hiele, 1986). Van Hiele emphasises that this theory is applicable not only to geometry, but to all sciences, even teaching (van Hiele, 1986, p. 86).

Levels of thought are named according to activities specific for the level (van Hiele, 1986, p. 53), while other authors also use other terms, given in brackets (Baranović, 2015):

- Level 1 is a visual level, as shapes are identified based on their appearance, and not based on their properties (Recognition).
- Level 2 is a descriptive level, as shapes are investigated, analysed and described based on their properties (Analysis).
- Level 3 is a theoretical level with logical relations, as logical relationships are formed between properties of a certain shape or between properties of different shapes. Only from this level it is possible to construct hierarchical classification of shapes (Informal deduction).
- Level 4 is a formal logic level, as it entails investigating the laws of logic, role of definitions, axioms, theorems and proofs within the deductive axiomatic system as a whole (Formal deduction).
- Level 5 is a level of nature of logical laws, the level at which different axiomatic systems are investigated and mutually compared, using pure mathematical language (Rigor).

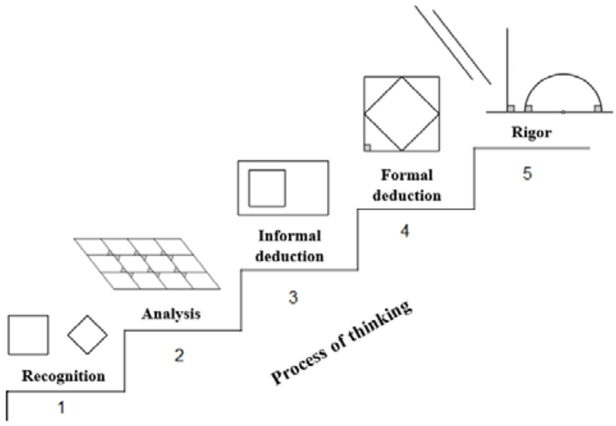


Figure 1: The van Hiele model

Bearing in mind that the focus of this paper is geometry, we consider the van Hiele model through the development of geometrical reasoning (Figure 1). This means that in learning Euclidean geometry, with the aim of achieving a certain level of geometrical reasoning, learning process should begin by recognising geometrical shapes based on their appearance (Level 1), followed by analysis of observed shapes, identifying their properties and naming them accordingly, but still without forming relations between them (Level 2). Afterwards, relationships between the properties of one shape or between shapes are recognised, meaningful definitions are formed, theorems are derived and conclusions are substantiated, identifying necessary and sufficient conditions for a certain concept (Level 3). Only after mastering the third level, progress can be made to independent derivation of formal proofs and understanding of necessity and justification of axiomatic construction of mathematical theory (Level 4). The pinnacle of learning and development of geometrical reasoning (Level 5) is understanding and comparison to other, non-Euclidean geometries (Burger & Shaughnessy, 1986; Crowely, 1987; Mason 2002; van Hiele, 1986).

This paper does not consider phases of learning, and only those characteristics of the five-level model which are used in interpreting the results will be taken into consideration. Firstly, progression through levels must be sequential, without omitting levels. In order to successfully perform at a certain level, the main precondition is to acquire knowledge, skills and language of previous levels. Secondly, if one subject area is studied in more detail than other areas, levels of thought can be different for those subject areas. Thirdly, advancement from one level to another or lack of advancement does not depend on age or maturity level of learners, but is more dependent on the modes of learning and teaching (Abdullah & Zakaria, 2013; Burger & Shaughnessy, 1986; Baranović, 2015; Crowely, 1987; Usiskin 1982, van Hiele, 1986).

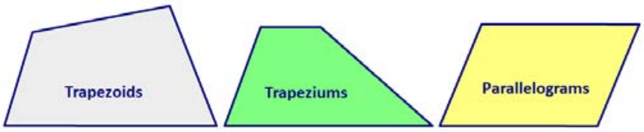
The van Hiele theory has been used by a number of mathematical education researchers for more than half a century, as a framework for interpreting advancement in development of (geometrical) reasoning of learners, emphasising the following: in order to be able to understand the hierarchical classification of quadrilaterals, students must reach at least the third level of geometrical reasoning.

Quadrilateral classification

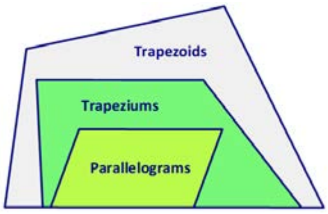
Classification of a concept is performed on the basis of a selected criterion, where concept definition plays a very important role (Fujita & Jones, 2007). Generally speaking, concept definition is not uniform, but a matter of agreement and can be performed in different manners, thereby creating different classifications of concepts (De Villiers, 2009; Kozakli Ulger & Tapan Broutin, 2017).

Depending on the manner of creating relations between concepts, definition can be exclusive or inclusive. Exclusive definition classifies concepts into sets which are mutually disjunctive; therefore, we call them partition classifications. Inclusive definition classifies concepts into sets which are not mutually disjunctive, and individual sets can have subsets (special cases), therefore we call them hierarchical classifications (Josefsson, 2016; De Villiers, 1994).

For example, in order to classify quadrilaterals based on properties of parallel sides, we consider shapes without parallel sides (trapezoids), shapes with one pair of parallel sides (trapeziums) and shapes with two pairs of parallel sides (parallelograms). If we consider that the trapezium is a quadrilateral with exactly one pair of parallel sides, trapezium is defined exclusively and trapeziums and parallelograms are disjunctive classes, i.e. neither parallelograms are trapeziums, nor are trapeziums parallelograms (Figure 2(a)). If we consider that the trapezium is a quadrilateral with (at least) one pair of parallel sides, then the trapezium is defined inclusively and the set of parallelograms is a subset of the set of trapeziums, i.e. all parallelograms are trapeziums, whereas all trapeziums are not parallelograms (Figure 2(b)). Although the criterion of parallel sides is most often used for classification of quadrilaterals, various other criteria can be used (Graumann, 2005, p. 191).



(a)



(b)

Figure 2: Partition and hierarchical classification of quadrilaterals

When certain properties of a concept are chosen for formulating a definition, all other properties of that concept are logically derived, expressed by theorems and proved (De Villiers, 2009). In that process, inclusive definitions are more practical because all assertions made and proved for a specific concept are valid for all special cases. In exclusive definitions every assertion must be formulated separately and proved for each concept (Josefsson, 2016; De Villiers, 1994). For example, if we use inclusive definition, one can simply derive the formula for area of parallelogram from the formula for area of trapezium by inserting equal bases, whereas if we use exclusive definition both formulas must be derived separately, independently of each other.

At the beginning of mathematical education, exclusive definitions are more appropriate than inclusive, until learners master the first two levels of thought. However, through the process of defining, students should be gradually guided to inclusive relations based on which they can understand hierarchical classification, which is far more functional than partition classification (De Villiers, 1994).

For each geometrical shape, we can consider its Concept Image (mental pictures, properties and processes associated with the concept) and Concept Definition (formal verbal description of concept), which together form a Figural Concept. Students usually have personal figural concepts which differ from formal ones (Fujita & Jones, 2007; Kozakli Ulger & Tapan Broutin, 2017). Research shows that pictures from Concept Image become strong prototypes which dominate the definition process and problem solving, creating difficulties in transition to inclusive relations and hierarchical classification (Erez & Yerushalmy, 2006; Fujita & Jones, 2007; Kozakli Ulger & Tapan Broutin, 2017). For example, if we constantly use visual representation of parallelogram as in Figure 2, for students this becomes a prototype based on which they build their Figural Concept. With such prototype of parallelogram, learners find it difficult to accept the possibility that rectangles, squares and rhombuses are also parallelograms.

A number of educational research papers confirm that the skill of defining geometrical concepts has an important role in learning geometry and development of geometrical reasoning. Therefore, students should be allowed first to create their own definitions, using their own language, corresponding to the level of thought they had reached, even if they are wrong. The process of defining should end in agreement, choosing one common concept definition. The other properties of a concept should be described by theorems and proved (e.g. De Villiers, 2009; Kozakli Ulger & Tapan Broutin, 2017).

In addition to selecting a unified definition of a concept, it is important to consistently use the same type of definition, in our case either exclusive or inclusive, avoiding using them interchangeably. If we allow that a square is a special case of rectangle, trapezium cannot be defined exclusively, which is a common case in mathematical practice (Josefsson, 2016). Although we cannot consider any definition which is not contradictory to some other definition or theorem to be faulty, still some definitions are better than others (De Villiers, 1994).

Textbooks in Croatian educational system use an inclusive definition for defining parallelogram, therefore rhombus, rectangle and square are considered as special types of parallelogram and this approach is mostly unified. However, there are differences in approach to trapezium and kite, and in case of trapezium both types of definitions are included in textbooks, with more examples of exclusive than inclusive definitions, as confirmed by results of research by Josefsson.

3. Method

This research was conducted as a part of a wider research of primary education students' knowledge, understanding and correlation of geometrical concepts and the influence of visual-spatial skills on geometry learning outcomes. This paper focuses on knowledge on quadrilaterals, specifically: square, rectangle, rhombus, parallelogram, trapezium and kite. The aims of the research are as follows:

a) Establish *which knowledge* students have on quadrilaterals, i.e. are they able to name the shapes and describe their properties based on visual representation.
b) Establish *in what manner they create relationships* among quadrilaterals: i.e. do they construct partition or hierarchical classification of quadrilaterals.
c) Establish *dominant traits* in classification of quadrilaterals: visual representation or knowledge they have on quadrilaterals.

Research was conducted as a part of the project Cognitive development and geometry learning outcomes *of Primary Education students*[5], related to doctoral research on development of visual-spatial skills and geometric reasoning of primary education students based on the method of direct observation and van Hiele theory.

[5] Research was partly funded from programme resources of the Faculty of Humanities and Social Sciences in Split, within the framework of KRIUGS 2015 project.

Participants

Research was conducted during the summer academic term with 3rd-year undergraduate students, before the course *Mathematics 2*, covering topics related to plane and solid geometry. The sample comprised of 36 out of 50 (72%) enrolled students, with average age 21.8. Students who declined to participate in the research considered that their existing knowledge of geometry was poor and they *did not want to embarrass themselves*.

Students enrolled in University of Split primary education study programme come from secondary schools with different study programmes (college preparatory, vocational, art and design), mostly located in Dalmatia (south of Croatia, inland, coast, islands). Admission requirement for the study programme is B level (lower level) at national secondary school leaving examinations. As a result, it is likely that the students' existing levels of knowledge is poor and inconsistent.

Students do not take courses in mathematics at the 1st and 2nd year of undergraduate study programme. In the winter academic term of the 3rd year, students take *Mathematics 1* (topics related to logic, sets, functions and hierarchy of numbers - from natural to real numbers). Considering that the research was conducted before geometry classes and without any preparations, after 2.5-year-break, it can be considered that obtained results reflect students' permanent knowledge of geometry, acquired after 12 years of mathematics education.

Instrument and data collection procedure

For the purpose of this research, a three-part questionnaire was used, with 30-minute time limit. The first part included an open-ended task, asking students to assign a number to any shape on square dotted grid (Figure 3), write the name of each shape with the corresponding number in the table below and all properties of that shape, which they already know or can recognise from the picture.

The purpose of this task was twofold. Firstly, students were given the opportunity to recall what they learned about basic properties of square, rectangle, rhombus, parallelogram, trapezium and kite. Shapes were placed on a grid to allow direct observation. Secondly, qualitative analysis of their answers can provide insight into existing knowledge of quadrilaterals and give answer to research question a), while also substantiating conclusions made from answers in the remaining part of the questionnaire.

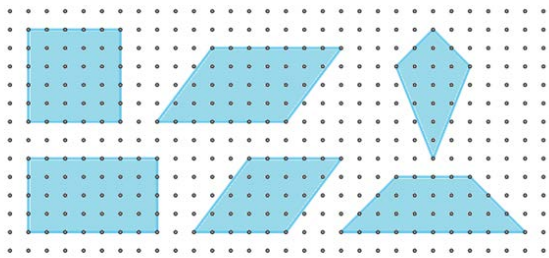


Figure 3: Quadrilaterals in open-ended task

In the second part of the questionnaire, the task was to classify quadrilaterals which are visually represented and numbered from 1 to 20 (Figure 4), by completing the table provided below. The quadrilateral classes are written in the table, in the following order: parallelogram, rectangle, square, rhombus, trapezium and kite. Considering that the task has multiple solutions, there is a remark explaining that one quadrilateral can belong to several classes.

Shapes are given in their standard position and form (e.g. 7, 13, 16), in standard form but in non-standard position (e.g. 9, 18, 19) and in non-standard position and form (e.g. 4, 8, 17). The purpose of this section of the questionnaire was to investigate to what extent the position and form of a shape influence recognition and classification of a specific type of quadrilateral. Analysis of responses in this section can provide grounds for establishing which relationship is constructed between classes based on visual representation: partition or hierarchical.

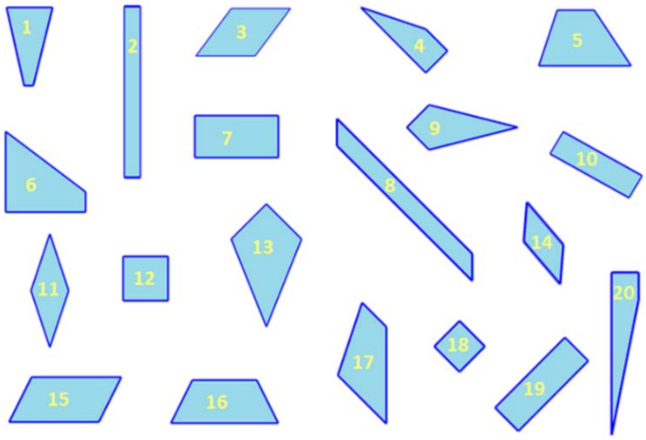


Figure 4: Picture problem with quadrilaterals

The third part of the questionnaire contains 20 simple statements that need to be completed, choosing one of three provided answers: always, sometimes or never (Figure 5). The statements describe all bilateral relations between observed quadrilaterals (except kite).

The purpose of this task was to investigate in what manner and to what extent students establish a theoretical connection between quadrilaterals, without being influenced by visual representation. Analysis of responses in this section shows different levels of connections and indicates the ones that have even not been established. By combining the results obtained in the second and third part of the questionnaire, research questions b) and c) can be answered.

A square is ______________ a rectangle.	A rectangle is ________________a parallelogram.
A parallelogram is _________________a rhombus.	A rhombus is ________________a trapezium.
A trapezium is ________________ a square.	A square is ______________a parallelogram.
A parallelogram is _________________ a trapezium.	A trapezium is ________________a rectangle.
A rectangle is ________________a rhombus.	A rhombus is ___________________a square.
A square is ______________ a trapezium.	A trapezium is ________________a rhombus.
A rhombus is ___________________a parallelogram.	A parallelogram is ______________a rectangle.
A rectangle is ________________a square.	A square is ______________a rhombus.
A rhombus is ___________________a rectangle.	A rectangle is ________________a trapezium.
A trapezium is ________________a parallelogram.	A parallelogram is ______________a square.

Figure 5: Questions in the third section

The idea for developing this instrument evolved from the research of Žilková (2015) and Jozić (2010), with the author's personal teaching experience playing a pivotal role.

Results and discussion

Results of the research are analysed and discussed in relation to three research questions concerning the primary education students' knowledge on quadrilaterals (square, rectangle, rhombus, parallelogram, trapezium and kite):

(a) Are students able to name the shapes and describe their properties based on visual representation?
(b) Which relationship do the students establish among quadrilaterals: partition or hierarchical classification?
(c) What are dominant traits in students' classification of quadrilaterals: visual representation or knowledge they have on quadrilaterals?

Students' knowledge considering quadrilaterals

Qualitative analysis of students' responses in the first section of the questionnaire shows that students have no difficulties in identifying shapes based on visual representation, except three students (8.33%). One student failed to name a kite, one student named a kite as rhombus, not identifying rhombus at all, and one student named a parallelogram as rhombus, failing to identify rhombus and kite.

Student mostly describe properties related to sides, angles, diagonals, perimeter and area (Table 1). It is apparent that the students focused their attention to sides and angles, disregarding other properties of shapes, especially properties of trapezium and kite. More detailed analysis can be used to establish which properties present difficulties, either in the process of identifying or making correlations (Table 2).

1st part	Sides	Angles	Diagonals	Perimeter	Area	Various	Blank
Square	100%	86%	53%	39%	44%	28%	-
Rectangle	94%	81%	36%	56%	53%	14%	-
Parallelogram	100%	67%	36%	22%	17%	28%	-
Rhombus	97%	69%	39%	17%	17%	17%	3%
Trapezium	81%	50%	14%	3%	3%	19%	6%
Kite	81%	42%	14%	6%	6%	17%	11%

Table 1: Frequency of properties per shape

All things considered, students have poor level of knowledge of properties of quadrilaterals, with varying results and some basic misconceptions. According to main characteristics in formal definitions used in Croatian educational system, it is evident that 31 (86%) students recognise those properties in squares (equal sides, equal angles), 29 (81%) in rectangles (equal opposite sides and all angles), 27 (75%) in parallelograms (parallel opposite sides), 30 (83%) in rhombuses (all sides equal), 19 (53%) in trapeziums (one pair of parallel sides) and 23 (64%) in kites (two pairs of adjacent sides equal). Therefore, it can be concluded that around 80% of students are relatively well-acquainted with the notion of square, rectangle, parallelogram and rhombus, while slightly over 50% of students are acquainted with the notion of trapezium and kite.

1st part	Square	Rectangle	Parallelogram	Rhombus	Trapezium	Kite
Equal sides	all sides (36; 100%)	opposite (31; 86%)	opposite (35; 97%)	all sides (30; 83%)	legs (24; 67%)	two pairs adjacent (23; 64%)
Equal angles	all angles (31; 86%)	all angles (29; 81%)	opposite (10; 28%)	opposite (8; 22%)	angles on base (7; 19%)	one pair opposite (7; 19%)
Parallel sides	two pairs (7; 19%)	two pairs (15; 42%)	two pairs (27; 75%)	two pairs (3; 8%)	one pair (19; 53%)	-
Equal diagonals	yes (15; 42%)	yes (13; 36%)	no (11; 31%)	no (10; 28%)	yes (5; 14%)	no (2; 6%)
Perpendicular diagonals	yes (9; 25%)	no (1; 3%)	no (1; 3%)	yes (6; 17%)	no (1; 3%)	yes (5; 14%)
Perimeter	$P = 4a$ (14; 39%)	$P = 2a + 2b$ (20; 56%)	$P = 2a + 2b$ (8; 22%)	$P = 4a$ (6; 17%)	$P = a + 2b + c$ (1; 3%)	$P = 2a + 2b$ (2; 11%)
Area	$A = a^2$ (16; 44%)	$A = ab$ (19; 53%)	$\mathbf{A = ab}$ (6; 17%)	$\mathbf{A = a^2}$ (6; 17%)	sum of areas (1; 3%)	sum of areas (1; 3%)

Table 2: Frequency of responses by individual properties

However, looking into other properties, the students' knowledge is limited and tends to decrease starting from square and rectangle towards trapezium and kite, especially concerning angles and diagonals. For example, although 19 (53%) students responded that trapezium has one pair of parallel sides, 14 (39%) students added that the sides are of different length, thereby immediately excluding parallelograms as a subset of trapeziums. On the other hand, although the picture shows isosceles trapezium, this is mentioned by 24 (67%) students (isosceles trapezium is named by only 8 students; 22%), equal angles on base is mentioned by only 7 (19%), and equal diagonals by only 5 (14%) students.

Surprisingly small number of students wrote formulas for calculating perimeter and area for square and rectangle, considering that these formulas are used since the 4th grade of primary school and represent a basis for the concept of area. On the other hand, few students who wrote the formulas for parallelogram and rhombus wrote an incorrect area formula (6 students; 17%). In addition, in formulas they use standard designations for side length, although they omit putting designations in pictures, which indicates that they learn formulas by heart. Incorrect formulas indicate misunderstanding of the concept of area.

Based on all results obtained in the first part of the questionnaire, it can be concluded that the basic knowledge on quadrilaterals is quite poor and varies from one participant to another, with a declining trend starting from square towards trapezium and kite, which partly corresponds to the results obtained by other researchers (e.g. Fujita & Jones, 2007, Žilková, 2015). Furthermore, it can be concluded that most students advanced to the first level of thought according to the van Hiele model, while a number of students do not have sufficient or sound knowledge for the second level.

Identification and relationships among quadrilaterals

The students' answers in the second part of the questionnaire indicate that only some shapes were identified without any problems, as students encountered difficulties both in identifying and classifying the rest of the shapes (Table 3).

2nd part	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11	F12	F13	F14	F15	F16	F17	F18	F19	F20
Parallelogram	0,03	0,28	0,53				0,25	0,97		0,25	0,42	0,22	0,03	0,86	0,94	0,03	0,06	0,19	0,22	
Rectangle		**1,00**	0,06				**1,00**	0,08		0,97	0,06	0,44		0,03	0,11	0,03		0,47	0,97	
Square		0,11	0,11				0,11			0,11	0,08	**1,00**		0,03				**1,00**	0,11	
Rhombus			0,83					0,06	0,03		0,97	0,14	0,03	0,22	0,03		0,03	0,22		
Trapezium	0,89	0	0,03	0,58	0,92	0,69	0	0		0	0	0		0	0	0,97	0,64	0	0	0,58
Kite	0,08		0,03	0,28	0,03	0,22			0,97		0,14	0,03	0,92				0,08	0,03		0,33

Table 3: Frequency of responses by individual figures

All of the students recognised squares in figures F12 and F18 and rectangles in figures F2 and F7. In figures F10 and F19 all but one student recognised rectangles, therefore it can be said that students recognise square and rectangle in spite of position and form, i.e. for these shapes they have reached the first level of geometric reasoning according to van Hiele. However, this is not the case for all quadrilaterals.

Establishing relationships between square, rectangle and other quadrilaterals varies greatly, and in some cases is very poor or non-existent. While almost half of the students categorize squares under rectangles, only around one-fifth of them place squares in sets of parallelograms and rhombuses, and one-fourth of students classify rectangles as parallelograms. Students fail to establish inclusive relationships of trapezium, i.e. trapeziums are placed in disjunctive sets in relation to square and rectangle. Moreover, kites are placed in disjunctive sets in relation to squares by all but one student.

Unexpectedly poor results of classification were obtained for parallelograms (F8, F14, F15) and rhombuses (F3, F11). For some reason 6 (17%) of students did not recognise rhombus in F3, and 2 (11%) did not recognise parallelogram in F15, although the shapes were represented in standard position. All but one student recognised rhombus in F11 and parallelogram in F8, in spite of untypical position and form. Results for the relationship between rhombus and parallelogram are somewhat better (for F3 19 students, 53%, and for F11 15 students, 42%) than the results for the relationship of square and rectangle with parallelogram, which corresponds to findings of other researchers (Fujita & Jones, 2007; Žilková, 2015). Rhombuses and parallelograms were not classified under trapezium (except by one student), and only 5 (14%) of students placed rhombuses in the class of kites. Therefore, the students definitely do not recognise parallelograms as trapeziums.

There is a noticeable influence of different position and form in case of trapeziums. While the majority of students recognised a trapezium in F5 and F16 and a surprisingly high percentage of participants recognised a trapezium in F1 (32; 89%), only around 60% of students recognised a trapezium in F4, F6, F17 and F20. The cause could lie in teaching method and instruction mode, but also the time period devoted to related subject matter (van Hiele, 1986). In Croatian educational system, trapezium is part of the final module of 6th grade curriculum and it is possible that not enough time was devoted to this topic.

In addition to poorly established or non-existent relationships, it is worrying that some students incorrectly categorized rhombuses as squares, parallelograms as rectangles (e.g. 8 (22%) in F14), trapeziums as parallelograms, rectangles and kites (e.g. 12 (33%) in F20, 10 (28%) in F4 and 8 (22%) in F6, which are right-angled trapeziums), and kites as rhombuses. It could be said that their thinking is quite chaotic and they do not trust what they see.

Based on the results obtained in the second part of the questionnaire and taking into consideration the results from the first part, conclusion can be made that students who have relatively good knowledge of properties of quadrilaterals are able to form relationships between quadrilaterals, whereas those students who are not so familiar with properties of quadrilaterals have difficulties establishing relationships between the shapes. The results clearly follow the van Hiele model, stating that learners cannot advance to level 3 until they have mastered levels 1 and 2. Finally, students only partially establish hierarchical relationship between parallelograms, whereas trapezium and kite remain in partition relationship, which is consistent with the results of other researchers (e.g. Josefsson, 2016; Fujita & Jones, 2007; Kozakli Ulger & Tapan Broutin, 2017; Žilková, 2015).

Image versus knowledge

In the third section, which includes establishing theoretical relations between quadrilaterals by completing sentences, kite is omitted. The theoretical concept of a kite as quadrilateral is not included in the Croatian school curriculum and textbooks, but is considered only in some isolated examples or tasks.

The answers given by students in the third part of the questionnaire (Table 4) indicate certain discrepancy in relation to answers given in the second part. Although 29 (81%) of students confirmed that a rhombus is always a parallelogram, in the second part some of these students did not classify rhombuses as parallelograms, i.e. this was not done by 12/30 students (40%) in F3, and 16/35 students (46%) in F11. Also, although 28 (78%) of students confirmed that a square is always a rectangle, still 13/28 (46%) of students did not classify squares in F12 and F18 as rectangles. The relationship 'a square is always a parallelogram' shows similar results. Therefore, although students consider theoretical assertions as true, the image in front of them hinders the classification process.

On the other hand, inability to establish relationships between separate parallelograms and trapeziums is confirmed by classification of shapes in the second part of the questionnaire. The students have firmly accepted partition relationship, as confirmed by third column in Table 4, with high percentage of answers: 30 (83%) state that a square is never a trapezium, 31 (86%) state that a rectangle is never a trapezium, 27 (75%) state that a parallelogram is never a trapezium and 26 (72%) state that a rhombus is never a trapezium. These findings are also in line with results of previous research, stating that constructing and understanding hierarchical classification of quadrilaterals is quite difficult for many learners (e.g. Fujita & Jones, 2007; De Villiers, 2010).

3rd part	A	S	N	Relationship
Square	78%	0%	22%	Rectangle
	31%	14%	53%	Parallelogram
	14%	17%	64%	Rhombus
	3%	14%	83%	Trapezium
Rectangle	17%	58%	25%	Square
	33%	14%	50%	Parallelogram
	3%	14%	83%	Rhombus
	0%	6%	86%	Trapezium
Parallelogram	3%	56%	36%	Square
	6%	44%	44%	Rectangle
	3%	83%	14%	Rhombus
	0%	22%	75%	Trapezium
Rhombus	17%	14%	67%	Square
	0%	17%	83%	Rectangle
	81%	6%	11%	Parallelogram
	6%	14%	72%	Trapezium
Trapezium	3%	8%	89%	Square
	6%	19%	72%	Rectangle
	3%	19%	78%	Parallelogram
	3%	17%	75%	Rhombus

Table 4: Frequency of responses in establishing relationships

Based on the results obtained in the third part of the questionnaire and taking into consideration the results from the second part, conclusion can be made that students establish firmer relationships between parallelograms in theory, rather than based on images of individual shapes. This means that the image interferes with the classification process, as confirmed by other researchers. The images are dominant in attempts to define a quadrilateral, as well as in solving problems with quadrilaterals. Many students seem to struggle between the image and knowledge, instead of using the image to extend knowledge or as part of knowledge of quadrilaterals.

As far as establishing relationships between quadrilaterals is concerned, conclusion can be made that students establish relatively weak relationships between parallelograms, considering that square, rectangle and rhombus are studied as special types of parallelograms throughout primary and secondary education. Failure to establish relationships between individual parallelograms and trapezium is therefore expected, considering the followed curriculum.

Finally, students are better acquainted with partition classification – in spite of constructed hierarchical relationships, many students have great difficulties with hierarchical classification of individual quadrilaterals. The reason for this can be found in the first section of the questionnaire, which shows that many students have poor knowledge of properties of quadrilaterals and therefore they do not have proper basis for constructing relationships. Many students have not yet mastered the second level of thought and it cannot be expected that they can create inclusion relationships. Considering that the average age of students is 21.8, the findings confirm van Hiele's claim that the development of geometrical reasoning is not age-dependant but is a consequence of learning and teaching.

4. Conclusion

In this paper, one part of research conducted with third year undergraduate students of Primary Education at the University of Split Faculty of Humanities and Social Sciences was presented. The aim of the research was to establish the scope of students' knowledge of quadrilaterals, the manner in which they establish relationships among quadrilaterals and the dominant factor of this process, as well as to investigate possible misconceptions of these relationships. Obtained results indicate that students encounter difficulties in identifying quadrilaterals in non-standard positions or of non-standard shapes, have unexpectedly poor knowledge of quadrilaterals' properties and are not able to establish inclusion relationships, especially in the case of trapezium and kite. Results are comparable to results of other researchers (e.g. Armah, Cofie and Okpoti, 2018, De Villiers, 2009; Fujita and Jones, 2007; Erez & Yerushalmy, 2006; Gűnhan, 2014; Josefsson, 2016; Kozakli Ulger & Tapan Broutin, 2017).

Furthermore, the results show that many students did not attain the second level of thought in geometry according to van Hiele, and some did not even master the first level and are not able to construct inclusion relationships although the students are 20-year-olds. The findings are in line with the van Hiele theory, according to which the development of geometrical thought does not depend on age, but on learning and teaching, and that the third level of thought in geometry cannot be reached before mastering the previous two levels.

Although we cannot consider any type of definition (or classification) flawed, still some are better than others. Without understanding inclusion relations, learners cannot construct hierarchical classifications, which are more economical than the partition ones because once the properties of a certain class are stated and proved, e.g. parallelograms, the same properties do not have to be specifically stated or proved for subclasses, e.g. rectangles, because special cases inherit all the properties of a parent class. What needs to be avoided is partially using one and partially another type of definition because one part of relations in classification will be partition and one part hierarchical, which can cause confusion in students, as confirmed by the findings of this research. Still, at the beginning of education in math, children are able to understand only partition relationships; therefore teaching may begin with exclusive definitions, and gradually, for functional reasons, teachers should start building upon inclusive definitions.

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Predictors of the intention to use manipulatives
(One result of the research)

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The *use* of *manipulatives is a kind of intervention in a process of supporting and developing of mathematical thinking. Their use is a precondition for* effectiveness *and improving the quality of mathematics teaching. It is necessary that the intention of using manipulatives is completely appropriate and clarified to the teacher as to a subject who intervenes with this powerful tool which changes the circumstances in the learning environment. At the same time, manipulatives increase student participation level in the process of learning mathematics, strengthens their role as an actor of the educational process. The results of the research indicate that it is possible, based on the perception of familiarity, capabilities and a range of other parameters, to determine the circumstances and design recommendations for improving teacher competencies for the use of manipulatives. One of the results of the research is also the formula which, on the basis of the input of certain parameters, indicates the prediction of intentions to use manipulatives by a particular teacher.*

Keywords: manipulatives, developing mathematical thinking, teaching and learning mathematics.

Quality and effectiveness of educational work

There are numerous parameters that can describe the process of developing mathematical thinking. By describing the process, we create the preconditions for its monitoring and intervention, which should lead us to the projected goals or outcomes of the pedagogical work. *Quality* and *effectiveness* are certainly categories in a certain conjuncture and their understanding and establishment enables successful learning and teaching of mathematics and through mathematics.

If we characterize a set of characteristics of a particular entity that enable the satisfaction of the identified/expressed needs of the user as quality, then it is also a measure (level) of satisfaction of the requirements and expectations. Beneficiaries can be different instances - state, society, family, institution/school, individuals, parents, students, local government ... which increases the complexity of understanding and defining entity characteristics. Applied to educational work, some of the elements of quality in Serbia, defined by the *Law on the Education System Foundations* (*Sl. glasnik RS*, No. 88/17 and 27/18 - other law) are:

- principles of education,
- objectives of education,
- competences of students,

- learning environment,
- competences and professional development of teachers,

- …

Measuring, that is, determining the quality of educational work is done based on a unique framework, which is a document called the Rulebook on quality standards of work of an institution (Sl. glasnik RS – Prosvetni glasnik, No. 14/2018). This regulation defines several areas, the most important is the area of Teaching and learning. Some of the requirements in this area of quality are related to:

- Learning process management (teacher) - using different methods; directing/ leading interaction in the learning function; functional use of teaching materials.

- Adapting the work to the educational needs of students (teacher) - adjusting the mode of work and teaching materials.

- Acquiring knowledge, adopting values developing skills and competences (student) - collection, critical evaluation and analysis of ideas, responses and solutions; presentation of own ideas, original and creative solutions.

- Procedures in the function of learning (teacher and student) - setting yourself goals in learning.

- Creating opportunities for each student to be successful (teacher and student) - a variety of methods to motivate students; encouraging intellectual curiosity and free expressing an opinion; choices for students regarding the way (approach) the topic is treated, the type of work or the material.

Effectiveness is determined by "doing the right thing", that is, producing the intended/projected result, achieving the particular goal (outcome). If we focus on the concept of effectiveness of the organization, than we are talking about achieving the goals for which it was founded, because of which it exists - meeting the needs of consumers/users. *Measurability of effectiveness* is achieved by comparing inputs and process outputs (output and input difference). *Effectiveness of education* refers to the impact on *students' achievement*, that is, on *learning outcomes*.

The connection between the *effectiveness and the quality* of educational work should be derived from the connection between the process of teaching and learning and the results of that proces.

Measurability of the effectiveness of educational work, that is, teaching and learning, is associated with the contribution of various factors (learning environment, family, school, department, teachers, peers, organization of the learning process, learning methods and techniques, use of learning resources, learning content...).

Manipulatives and use of manipulatives in teaching and learning mathematics

Manipulatives are *actual* (physical, real) or *virtual* (computer/software programs, applets) objects through which (or: over which) the student works, using his/her sensory-motor skills, by empirically examining and exercising (determining) the acquired knowledge, abilities and skills; recognizes new circumstances and facts, connects and concludes, thus creating a new corpus of knowledge, abilities and skills

Using manipulatives, senso-motor activities coordinate with mental activities; the connection between the concrete and the abstract is created (Damjanović, 2008, 36).

The use of manipulatives allows students to actively participate in the process of school learning (teaching and learning) and, in a wider context, provide (or should provide) smooth (undisturbed) development of mathematical thinking.

The development of mathematical thinking goes through phases (stages) that need to be harmonized and synchronized, without making the barrier to the one that follows.

Manipulative materials are objects designed to represent explicitly and concretely mathematical ideas that are abstract. They have both visual and tactile appeal and can be manipulated by learners through hands-on experiences … actively manipulating these materials allows learners to develop a repertoire of images that can be used in the mental manipulation of abstract concepts (Moyer, 2001, 176). Manipulatives are a representation of something else, not of themselves (Uttal, Scudder and DeLoache, 1997, 49). Without instruction, students may treat manipulatives as interesting objects that have little or no connection to anything else (Uttal, Scudder and DeLoache, 1997, 51).

Teachers need to recognize and properly use manipulatives as dual representations of mathematical objects, ideas and concepts. Therefore, it is insisted that every manipulative has a clear purpose and a moment to use, that is to be used just at the right time in the right way. In vocabulary of management, of process management, it would be *just in time*.

In this way, the use of manipulatives opens space for *student participation* in the learning process, as the ontological determinant of teaching and educational process.

The use of manipulatives creates the preconditions for achieving the quality of educational work in the teaching of mathematics in terms of fulfilling the requirements and expectations from a unique quality framework and establishing effective pedagogical practice. Experience and practical operations over objects/representations improve a successful transition from concrete to formal and abstract mathematical thinking. For students, the conditions are created to adopt (internalize) the mathematical space of objects, ideas, concepts, relationships, and operations, and connect them into a unique whole of their cognitive habitus.

The use of manipulatives in teaching mathematics – research

The practice of teaching and learning mathematics using manipulatives in the first and second cycle of comprehensive/compulsory education in Serbia was researched (explored) for the purposes of the thesis/dissertation (Damjanović, 2016) and the results were published in a separate research report (Damjanović, Stamenković, Popović and Dimitrijević, 2015). The research involved mathematics teachers and examined their experiences, insights and attitudes regarding the use of manipulatives.

Some of the objectives (goals) of the research on the use of manipulatives in mathematics teaching relate to the realization of the insight (refer to gain insight on) into the uniformity in the use between schools, between the two cycles of education in the primary school, the ability of teachers to use (through initial education and later through professional training), the use of virtual manipulatives working with students with developmental and learning *disorders* (disabilities)...

The research sample consisted of 697 teachers of class and subject teaching from two school administrations, from three districts. These teachers teach in 127 primary schools, deployed in 90 settlements of different sizes and structures of the population, that is, of different socio-economic and cultural composition.

The questionnaire contained questions of an open type in order to accomplish a qualitative insight. The answers that were obtained could be classified into categories, so that quantitative data processing was achieved.

The statistics applied is common to examine the relationship of categorical variables and to compare the distribution of continuous variables.

After the results of the research are published, another search of the base was carried out, colloquially speaking, and subsequently certain regularities, i.e. connection of data were observed. On this occasion, a univariate and multivariate binary logistic regression was used and a unique result was obtained, in fact a model that makes it possible to evaluate/estimate whether a particular teacher uses a manipulative in the teaching and learning process of mathematics, or not.

χ^2-squared test shows that the use of manipulatives is related to school administration, whether the teacher has met with the concept of manipulatives during the course of education, to what extent did he/she learn manipulatives during his/her education, whether the teacher has met with the concept of manipulatives during the work experience and to what extent did he/she learn manipulatives during his/her work experience.

Univariate binary regression shows that the use of manipulatives is influenced by the school administration, whether the teacher has met with the concept of manipulatives during the course of education, to what extent did he/she learn manipulatives during his/her education, whether the teacher has met with the concept of manipulatives during the work experience and to what extent did he/she learn manipulatives during his/her work experience.

Multivariate binary regression shows that the use of manipulatives is influenced by whether the teacher has met with the concept of manipulatives during the course of education and to what extent did he/she learn manipulatives during his/her work experience. (Multivariate binary regression examines/estimates the simultaneous effect of variables on the dependent binary variable).

Using the multivariate binary regression, the *Manipulative* model was created.[6]

Model ***Manipulative***

$$\boldsymbol{Manipulativ = 100e^{sum}/(1 + e^{sum})}, \textit{ while it's,}$$

[6] The *Manipulative* model on this occasion will be presented without discussion, because the text in which it will be discussed in detail is being drafted. The authors of the presentation decided to inform the participants of the Conference about the *Manipulative* model in this way, in order to interest the expert and scientific public to continue to monitor the results/findings of this particular research.

$$sum = 1{,}992a + 1{,}336b - 2{,}238$$

b is the extent /level to which a teacher has met the manipulatives during the work experience, and ***a*** is the binary variable such that ***a = 0*** if the teacher has not met with the concept of manipulative during the course of education, and ***a = 1*** if he is.

The variable *Manipulative* is calculated for each teacher and gives the probability that the teacher uses the manipulatives. Finally, the variable *Manipulative* is a marker to separate teachers who use manipulatives and the ones who do not use manipulatives. In other words, if we know whether a teacher has met the concept of manipulatives during the course of his education, and if we know the extent to which manipulatives have been learned during the work experience, then we can pretty much evaluate (estimate) whether he uses manipulatives (depending on whether the value of the Manipulative marker is less than 0.935 or not).

Conclusion

The role of teachers in educational process is enhanced by good initial readiness and continuous improvement. This training of teachers refers to readiness for changes in the learning environment, changes in the patterns of designing the context of teaching and learning, as well as the acceptance of responsibility for the implications of didactic directives and pedagogical acts aimed at the development of students' *thinking*.

Knowledgeof human resources, its initial competences, and its willingness to respond to the expectations of different educational instances for the implementation of appropriate and desirable pedagogy is a requirement which achievement represents a measure of quality and effectiveness.

The presented model *Manipulative* demonstrates the ability/possibility to detect the correlation of parameters and thus to manage changes in the circumstances of the educational process, which are related to the quality and effectiveness of teacher work.

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Закон о основама система образовања и васпитања Републике Србије, *Službeni glasnik RS*, 88/17.

Закон о основама система образовања и васпитања Републике Србије, *Službeni glasnik RS*, 27/18.

PROGRAMME AT A GLANCE

Place Hotel M, Belgrade, Serbia

Date May, 10-11, 2019

FRIDAY, MAY 10	SATURDAY, MAY 11
9:30 - 10:00 Registration	9:30-11:00 Workshop Dr Patrick Barmby "Research on monitoring and evaluation of knowledge in mathematics teaching – from quantitative to qualitative research methods"
10:00 - 10:15 Welcome and Opening	
10:15 - 11:00 Invited speaker Jarmila Novotna, Professor at Charles University, Czech Republic “Bridging two worlds – cooperation between academics and teacher-researchers”	
11:00 – 11:45 Invited speaker Dr Snežana Lawrence, senior lecturer at Middlesex University, United Kingdom “Motivation in the learning of mathematics: mathematics education and the founding principle of history”	

FRIDAY, MAY 10	SATURDAY, MAY 11
11:45 – 12:15 Coffee Break	11:00-11:30 Coffee Break
12:15 - 13:00	11:30 – 14:00

Invited speaker Dr Patrick Barmby, Head of Research at *No more Marking*, United Kingdom "Using a variety of methods for mathematics education research"	Museum Visit
13:00-13:45 Discussion with plenary speakers	
14:00-15:00 Conference Lunch	14:00-15:00 Conference Lunch
15:00 – 16:30 Reports Session 1	15:00 – 16:30 Reports Session 3
16:30-17:00 Coffee Break	16:30-17:00 Coffee Break
17:00-18:30 Reports Session 2	17:00- 18:30 Reports Session 4
	18:30-19:00 Closing session

REPORTS SESSION 1	
Friday **May 10** **15:00–16:30**	**Dragana Stanojević, Branislav Randjelović and Aleksandra Rosić** Educational standards in mathematics for the end of secondary education – Analysis of students' achievements

	Alenka Lipovec and Jasmina Ferme Lower elementary grades student teachers reflecting self-performed mathematical lesson **Nebojša Ikodinović, Jasmina Milinković and Marek Svetlik** Problem posing based on outcomes **Vojislav Andrić and Vladimir Mićić** Poet Desanka Maksimović’s high school graduation exam

REPORTS SESSION 2	
Friday **May 10** **17:00–18:30**	**Aleksandar Milenković and Sladjana Dimitrijević** Advantages and disadvantages of heuristic teaching in relation to traditional teaching – the case of the parallelogram surface **Radomir Lončarević** A different approach to solving linear Diophantine equations. An experimental study on using multiple strategies to solve linear Diophantine equations **Mika Rakonjac and Jasmina Milinković** Problem solving strategy – a criterion for the assessment of conceptual understanding **Milena Marić and Vojislav Andrić** About one case of research on mathematical knowledge of students

REPORTS SESSION 3	
Saturday **May 11** **15:00–16:30**	**Sladjana Dimitrijević, Branislav Popović and Marija Stanić** Influence of the type of formulation of mathematical tasks on students' success in solving it

	Aneta Gacovska Barandovska Research approach in math teaching – arguments for and against **Radojko Damjanović, Dragić Banković and Branislav Popović** Predictors of the intention of using manipulatives **Marija Radojčić** University students' opinions about secondary school subjects and their attitudes toward mathematics

REPORTS SESSION 4	
Saturday **May 11** **17:00–18:30**	**Nives Baranović** Students' knowledge about quadrilaterals **Radoslav Božić** The application of modern technology in teaching and learning stereometry **Milica Nikolić, Sonja Orlić and Ljubica Oparnica** Influence of mathematics textbooks on student achievements assessed by SPUR

Photo memories

Invited speakers

Prof. RNDr. Jarmila Novotná, CSc
Department of Mathematics and Mathematics Education, Charles University, Prague, Czech Republic

Dr Snežana Lawrence

Department of Design Engineering & Mathematics, Faculty of Science and Technology, Middlesex University, London, United Kingdom

Dr Patrick Barmby

No More Marking

Left Zoran Kadelburg, Jasmina Milinković and Vojislav Andrić

Right Plenary Panel

Below Auditorium

Auditourium

Coffee break

On Kalemegdan terrace

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