ДРУШТВО МАТЕМАТИЧА СРБИЈЕ

АКРЕДИТОВАНИ ПРОГРАМ:

345.
ДРЖАВНИ СЕМИНАР О НАСТАВИ МАТЕМАТИКЕ И РАЧУНАРСТВА ДРУШТВА МАТЕМАТИЧАРА СРБИЈЕ

Компетенција: К1
Приоритети: 3

ТЕМА 5:

ТАНГЕНТНИ ЧЕТВОРОУГАО

РЕАЛИЗАТОР СЕМИНАРА:

ВОЈИСЛАВ ПЕТРОВИЋ

БЕОГРАД,
09. – 10. 02. 2019.

TANGENTNI ČETVOROUGLOVI

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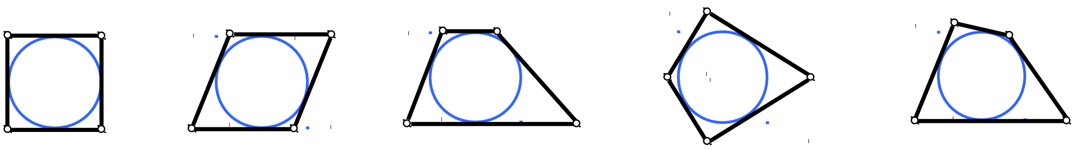
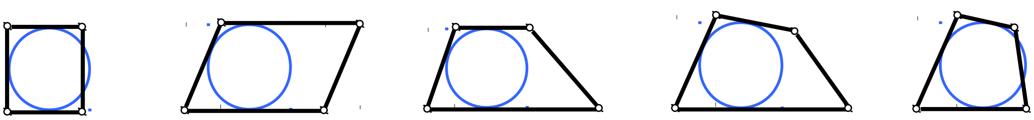
tangentan četvorougao - četvorougao u koji može da se upiše kružnica,
postoji kružnica koja dodiruje stranice četvorogla,
stranice četvorougla su tangente jedne kružnice

tangentni četvorouglovi



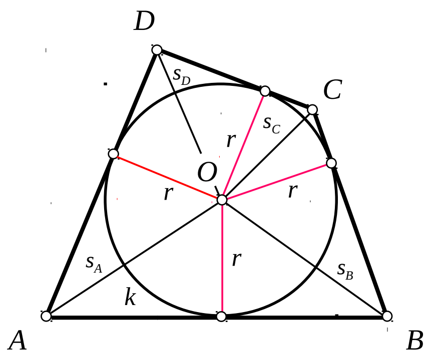
netangentni četvorouglovi

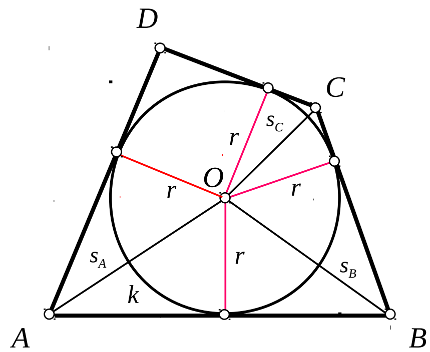


PITANJE. *Šta je potreban i dovoljan uslov da četvorougao bude tangentan?*

TEOREMA 1. *Četvorougao je tangentan ako i samo ako se simetrale četiri unutrašnja ugla seku u jednoj tački.*

■





TEOREMA 2. *Četvorougao je tangentan ako i samo ako se simetrale tri unutrašnja ugla seku u jednoj tački.*

■

TEOREMA 3. *Četvorougao ABCD je tangentan ako i samo je*

$$AB + CD = AD + BC.$$

Dokaz. (⇒) *ABCD tangentan*

k - upisana kružnica

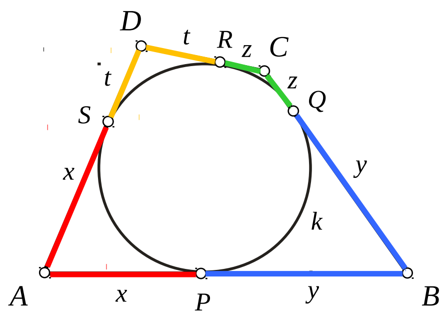
$AS = AP = x$ $\quad BP = BQ = y$

$CQ = CR = z$ $\quad DR = DS = t$

$AB + CD = (x + y) + (z + t)$

$AD + BC = (x + t) + (y + z)$

$\Rightarrow \ AB + CD = AD + BC = x + y + z + t$



(⇐) $AB + CD = AD + BC$ (1)

ABCD tangentni četvorougao

I *dokaz* (*školski*)

$s_A \cap s_B = \{O\}$ $k(O)$ - dodiruje DA, AB, BC

pretp. k ne dodiruje CD (2)

CD' - tangenta na k, $D' \in p(A, D)$

(2) ⇒ $\underline{A\text{-}D'\text{-}D}$ ∨ $A\text{-}D\text{-}D'$

$ABCD'$ tangentni

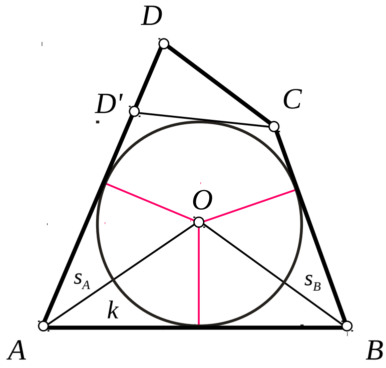
(⇒) ⇒ $AB + CD' = AD' + BC$ (3)

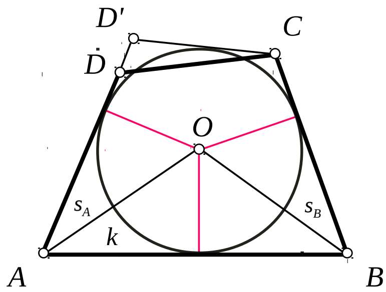
(1), (3) ⇒ $CD - CD' = AD - AD'$

⇒ $CD - CD' = DD'$ ↯ $\Delta CDD'$

slično za $A\text{-}D\text{-}D'$

⇒ (2) ne važi ⇒ k dodiruje i CD ⇒ $ABCD$ tangentni





$AB + CD = AD + BC$ (1)

II *dokaz*

$AB \geq AD \overset{(1)}{\Rightarrow} BC \geq CD$

(a) $AB = AD \overset{(1)}{\Rightarrow} BC = CD$

$\Delta ABC \cong \Delta ADC$ (SSS)

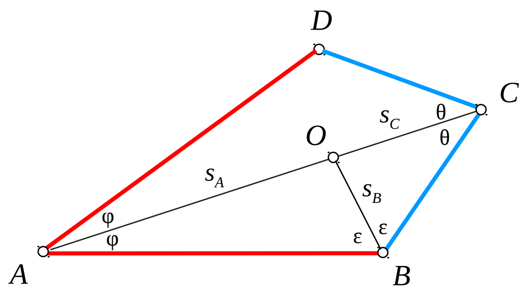
$\measuredangle BAC = \measuredangle DAC = \varphi$ (3)

$\measuredangle BCA = \measuredangle DCA = \theta$ (4)

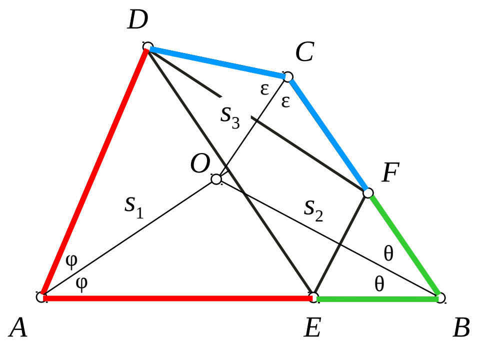
(3), (4) ⇒ $AC = s_A$, $CA = s_C$ (5)

$BO = s_B$, $O \in AC$ (6)

(5), (6) ⇒ $s_A \cap s_B \cap s_C = \{O\} \overset{T2}{\Rightarrow} ABCD$ tangenti



$$AB + CD = AD + BC \qquad (1)$$



(b) $AB > AD \overset{(1)}{\Rightarrow} BC > CD$

$A\text{-}E\text{-}B \ , \ AE = AD$

$B\text{-}F\text{-}C \ , \ CF = CD$

(1) $\Rightarrow \ AB - AD = BC - CD \ \Rightarrow BE = BF$

ΔAED - jednakokrak

$\Rightarrow s_A =$ sim. ED (7)

ΔBEF - jednakokrak

$\Rightarrow s_B =$ sim. EF (8)

ΔCDF - jednakokrak

$\Rightarrow s_C =$ sim. DF (9)

(7), (8), (9) $\Rightarrow s_1 \cap s_2 \cap s_3 = \{O\}$ - centar opisane kružn. ΔEFD

T 3 $\Rightarrow$ $ABCD$ - tangentan

III *dokaz*

Lema 1. *Ako kružnica upisana u ΔABC dodiruje stranice BC, CA, AB redom u tačkama K, L, M, tada je*

$$AL = AM = \tfrac{1}{2}(AB + AC - BC).$$

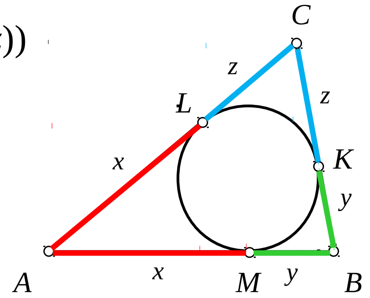
Dokaz leme 1.

$AL = AM = x \quad BM = BK = y \quad CK = CL = z$

$\frac{1}{2}(AB + AC - BC) = \frac{1}{2}((x + y) + (x + z) - (y + z))$

$= \frac{1}{2} \cdot 2x = x$

$= AL = AM$



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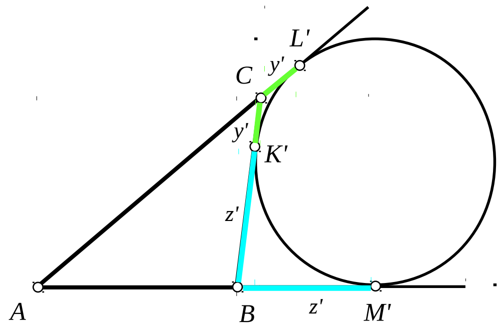
Lema 2. *Ako kružnica, spolja upisana u ΔABC, dodiruje stranicu BC u tački K' i produžetke stranica CA i AB redom u tačkama L' i M', tada je*

$$AM' = AL' = \frac{1}{2}(AB + AC + BC).$$

Dokaz leme 2. $\quad AM' = AL' \qquad (1)$

$BM' = BK' = z' \qquad CK' = CL' = y'$

$$\begin{aligned} AB + AC + BC &= AB + AC + (z' + y') \\ &= (AB + z') + (AC + y') \\ &= AM' + AL' \\ &\overset{(1)}{=} 2AM' \overset{(1)}{=} 2AL' \end{aligned}$$



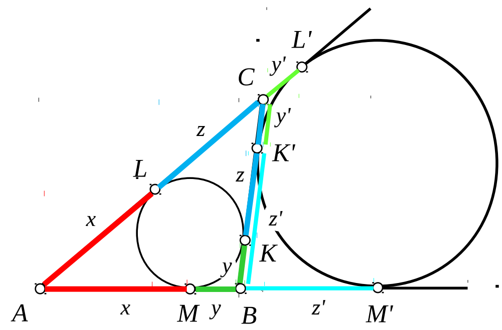
$\Rightarrow \quad AM' = AL' = \frac{1}{2}(AB + AC + BC).$

■

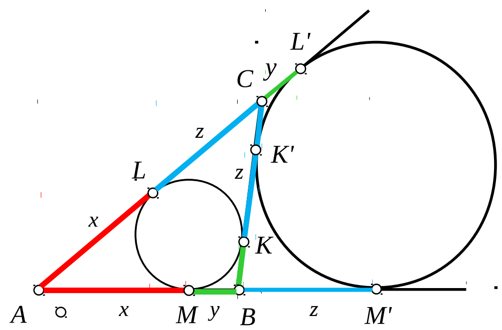
Напомена. $AM' = AL' = \frac{1}{2}(AB + AC + BC) = x + y + z$

$AM' = x + y + z' = x + y + z \Rightarrow z' = z$

$AL' = x + y' + z = x + y + z \Rightarrow y' = y$



$BC = MM' = LL' = y + z$



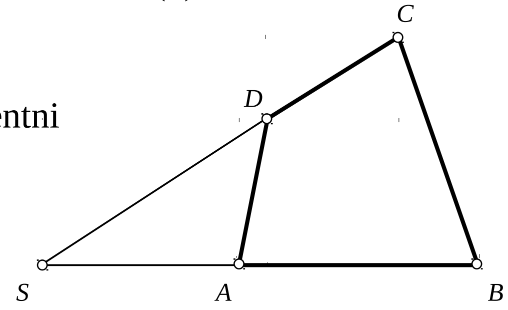
III dokaz - nastavak $\quad AB + CD = AD + BC \qquad (1)$

(a) *ABCD paralelogram*

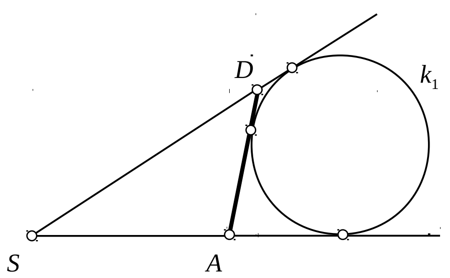
$(1) \Rightarrow$ *ABCD* romb $\Rightarrow$ *ABCD* tangentni

(b) *ABCD nije paralelogram*

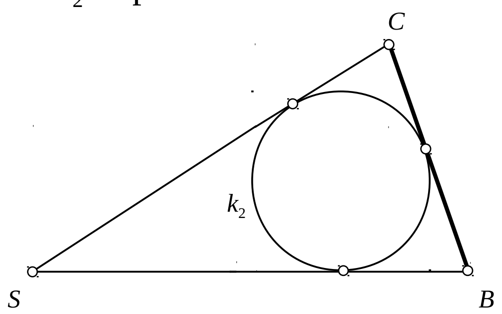
$AB \cap CD = \{S\}$



k_1 - spolja upisana u ΔSAD



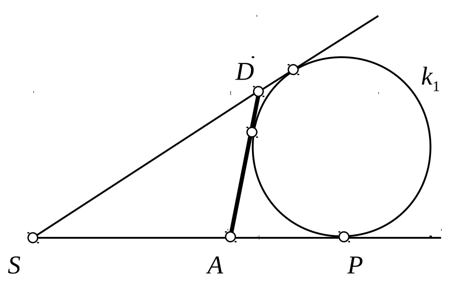
k_2 - upisana u ΔSBC

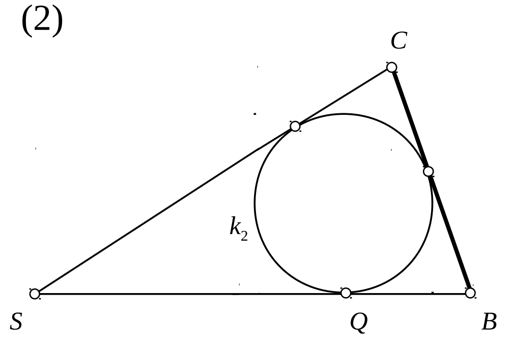


ABCD - tangentan $\Leftrightarrow k_1 \equiv k_2$

$AB + CD = AD + BC \qquad (1)$

ABCD - tangentan $\Leftrightarrow k_1 \equiv k_2 \qquad (2)$





$k_1 \equiv k_2 \Leftrightarrow SP = SQ \qquad (3)$

$$SP \overset{L2}{=} \tfrac{1}{2}\,(SA + SD + AD)$$

$$SQ \overset{L1}{=} \tfrac{1}{2}\,(SB + SC - BC) = \tfrac{1}{2}\,((SA + AB) + (SD + CD) - BC)$$

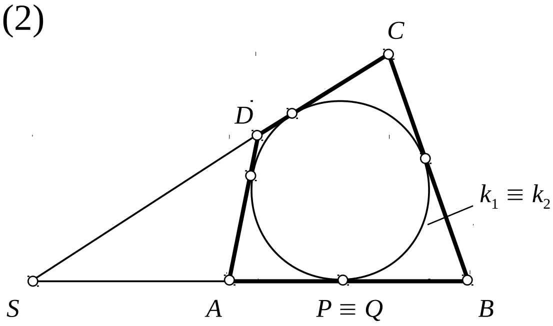
$$= \tfrac{1}{2}\,(SA + SD + (AB + CD - BC)) \overset{(1)}{=} \tfrac{1}{2}\,(SA + SD + AD)$$

$$= SP$$

$ABCD$ - tangentan $\Leftrightarrow$ $k_1 \equiv k_2$ (2)

$k_1 \equiv k_2$ $\Leftrightarrow$ $SP = SQ$ (3)

(2), (3) $\Rightarrow$ $ABCD$ - tangentan



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Primer 1. Ako se dijagonale tangentnog četvorougla seku u centru upisane kružnice, dokazati je taj četvorougao romb.

Rešenje.

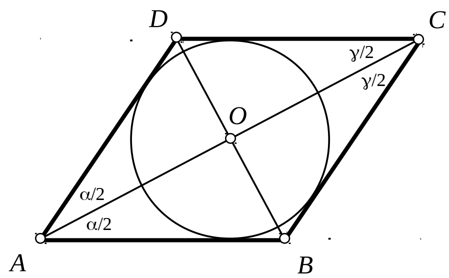
T 1 $\Rightarrow$ AC - simetrala $\sphericalangle A$ i $\sphericalangle C$

$\Rightarrow$ $\Delta ABC \cong \Delta ADC$ (USU)

$\Rightarrow$ $AB = AD$, $BC = DC$ (1)

slično $\Delta ABD \cong \Delta CBD$ (USU)

$\Rightarrow$ $AB = CB$, $AD = CD$ (2)



(1), (2) $\Rightarrow$ $AB = BC = CD = DA$ $\Rightarrow$ $ABCD$ - romb

■

Primer 2. Dokazati da je trapez tangentan ako i samo ako se kružnice konstruisane nad njegovim bočnim stranicama kao nad prečnicima dodiruju.

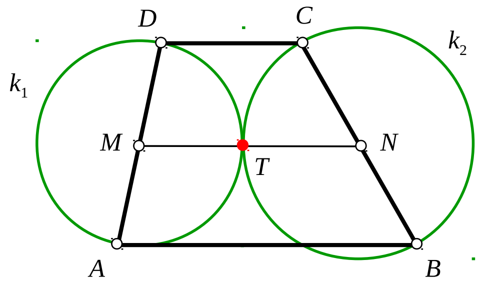
Rešenje. $(\Rightarrow)$ *ABCD - tangentni trapez*

$T\,3 \;\Rightarrow\; AB + CD = AD + BC \qquad (1)$

M - sred. AD $\qquad N$ - sred. BC

$MN = \frac{1}{2}(AB + CD) \overset{(1)}{=} \frac{1}{2}(AD + BC) \qquad (2)$

$k_1\left(M\,;\,\frac{1}{2}AD\right) \qquad k_2\left(N\,;\,\frac{1}{2}BC\right)$



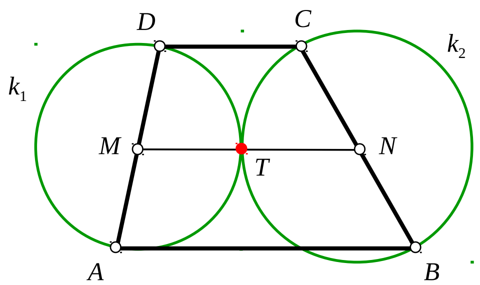
$(2) \;\Rightarrow\; k_1 \cap k_2 = \{T\}$

$(\Leftarrow)$ *ABCD - trapez, osnovice AB i CD*

M - sred. AD $\qquad$ *N - sred. BC* $\qquad k_1\left(M\,;\,\frac{1}{2}AD\right) \cap k_2\left(N\,;\,\frac{1}{2}BC\right) = \{T\}$

$$AB + CD = 2MN = 2(MT + NT)$$
$$= 2MA + 2NB$$
$$= AD + BC$$

$T\,3 \;\Rightarrow\; ABCD$ - tangentni



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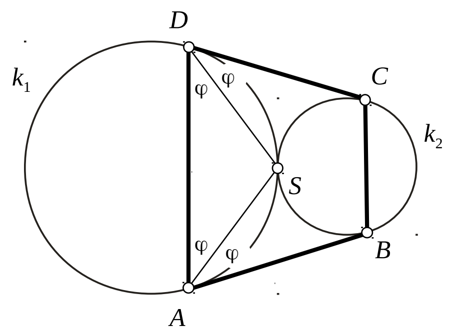
Primer 3. Kružnice k_1 i k_2 dodiruju se spolja. Dokazati da su tačke dodira zajedničkih spoljašnjih tangenti temena tangentnog četvorougla.

Rešenje. $k_1 \cap k_2 = \{S\}$

$ABCD$ - tangentni

I *varijanta*

$SA = SD \Rightarrow \angle SAD = \angle SDA = \varphi$ (1)



$\angle SAB = \angle SDA$ (2)

(ugao između tangente i tetive)

(1), (2) $\Rightarrow$ AS - sim. $\angle DAB$ (3)

slično DS - sim. $\angle ADC$ (4)

BS - sim. $\angle ABC$ (5)

CS - sim. $\angle BCD$ (6)

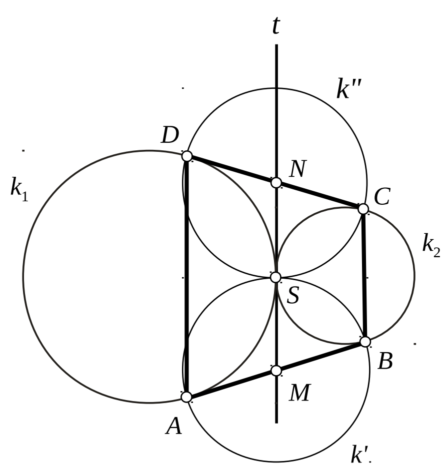
(3), (4), (5), (6), T 1 $\Rightarrow$ $ABCD$ - tangentni

II *varijanta*

$ABCD$ - trapez (jednakokraki) , $AD \,||\, BC$

t - zajednička tangenta k_1 i k_2 u S

$t \cap AB = \{M\}$, $t \cap AB = \{N\}$



$MA \overset{k_1}{=} MS \overset{k_2}{=} MB$ (6)

$ND \overset{k_1}{=} NS \overset{k_2}{=} NC$ (7)

$k' (AB) = k' (M ; MA)$

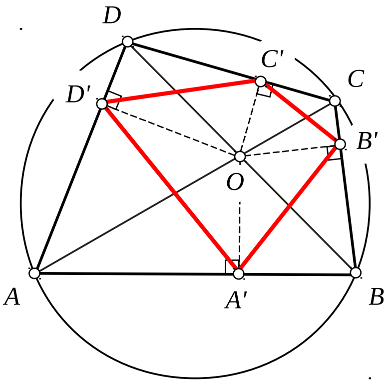
$k'' (CD) = k'' (N ; ND)$

(6), (7) $\Rightarrow$ $k' \cap k'' = \{S\}$

Pr. 2 $\Rightarrow$ $ABCD$ - tangentni

■

Primer 4. Neka se dijagonale tetivnog četvorougla *ABCD* seku u tački *O* i neka su *A'*, *B'*, *C'*, *D'* normalne projekcije tačke *O* na stranice *AB*, *BC*, *CD*, *DA*, redom. Dokazati da je četvorougao *A'B'C'D'* tangentni.



Rešenje. *O* - centar up. kružn. u *A'B'C'D'*

$\angle OA'D' = \varphi \qquad \angle OA'B' = \theta$

AA'OD' - tetivni ($\angle A' = \angle D' = 90^0$)

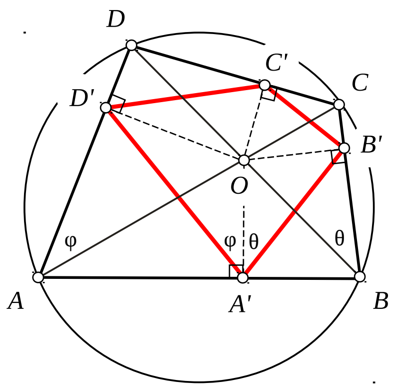
$\Rightarrow\ \varphi = \angle OA'D' = \angle OAD'$ (1)

nad lukom *OD'* kružn. $k_1(A, A', O, D')$

BB'OA' - tetivni ($\angle A' = \angle B' = 90^0$)

$\Rightarrow\ \theta = \angle OA'B' = \angle OBB'$ (2)

nad lukom *OB'* kružn. $k_2(B, B', O, A')$



$\angle CAD = \angle CBD$ (3)

nad lukom *CD* kružn. $k(A, B, C, D)$

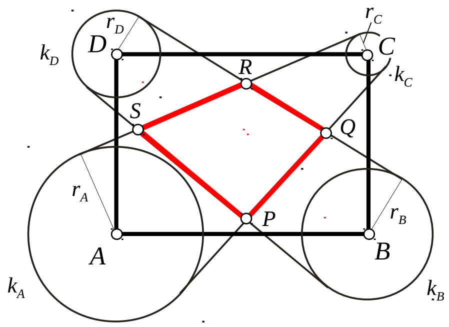
$\angle CAD \equiv \angle OAD'$, $\angle CBD \equiv \angle OBB'$ (4)

(1), (2), (3), (4) $\Rightarrow\ \varphi = \theta\ \Rightarrow$ *A'O* - sim. $\angle D'A'B'$

slično *B'O* - sim. $\angle A'B'C'$ *C'O* - sim. $\angle B'C'D'$ *D'O* - sim. $\angle C'D'A'$

T 1 $\Rightarrow$ *A'B'C'D'* - tangentni ■

Primer 5. Neka su k_A, k_B, k_C, k_D četiri disjunktne kružnice čiji su centri, tačke A, B, C, D, temena pravougaonika $ABCD$ kao na slici. Neka su poluprečnici kružnica redom r_A, r_B, r_C, r_D. Zajedničke spoljašnje tangente kružnica k_A i k_C seku se sa zajedničkim spoljašnjim tangentama kružnica k_B i k_D u tačkama P, Q, R, S. Ako je $r_A + r_C = r_B + r_D$, dokazati da je četvorougao $PQRS$ tangentni.



Rešenje. $r_A + r_C = r_B + r_D$ (1)

$AC \cap BD = \{O\}$

O - središte AC i BD

x', x'' - rast. O od PQ i RS

y', y'' - rast. O od QR i SP

$x' = x'' = \frac{1}{2}(r_A + r_C)$ (2)

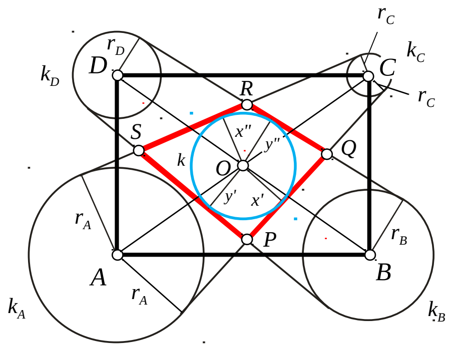
slično

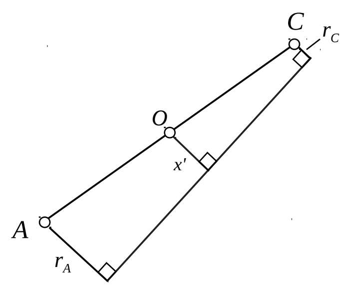
$y' = y'' = \frac{1}{2}(r_B + r_D)$ (3)

(1), (2), (3) $\Rightarrow$ $x' = x'' = y' = y''$

$\Rightarrow$ $k(O\,;x')$ - upisana u $PQRS$

$\Rightarrow$ $PQRS$ - tangentni





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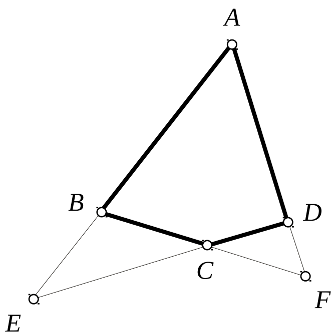
Primer 6. Dat je konveksan četvorougao *ABCD*. Prave *AB* i *CD* seku se u tački *E*, a prave *AD* i *BC* u tački *F* kao na slici. Dokazati da su sledeća tri tvrđenja ekvivalentna, tj. da svako povlači preostala dva:

(a) četvorougao *ABCD* je tangentni;

(b) $AE - AF = CE - CF$;

(c) $BE + BF = DE + DF$.

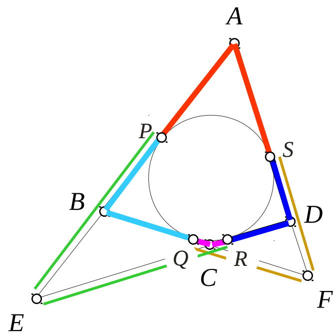


Rešenje. *ABCD* - *tangentni*

$AS = AP = a$ $\quad$ $BP = BQ = b$

$CQ = CR = c$ $\quad$ $DR = DS = d$

$EP = ER = e$ $\quad$ $FQ = FS = f$



(a) $\Rightarrow$ (b)

$AE - AF = (a + e) - (a + f) \quad = e - f$

$CE - CF = (e - c) - (f - c) \quad = e - f$

(a) $\Rightarrow$ (c)

$BE + BF = (e - b) + (b + f) \quad = e + f$

$DE + DF = (e + d) + (f - d) \quad = e + f$

(b) $\Rightarrow$ (a) $\quad AE - AF = CE - CF \qquad (1)$

$ABCD$ - tangentni

I *varijanta*

k - dodiruje AB, BC, CD

1° k dodiruje $DA \Rightarrow ABCD$ - tangentni ✓

2° k ne dodiruje DA

$A'D'$ - tangenta k i $A'D' \parallel AD$

$A'D' \cap CF = \{F'\} \qquad F'G \parallel AB$

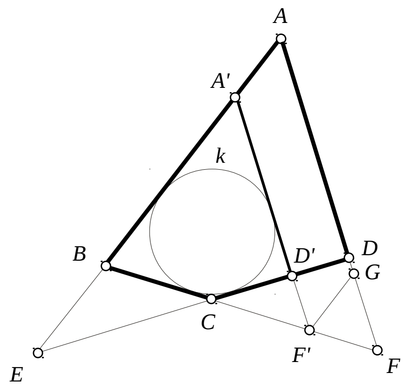
$A'BCD'$ - tangentni

$((a) \Rightarrow (b)) \Rightarrow A'E - A'F' = CE - CF' \qquad (2)$

$(1) - (2) \Rightarrow AE - A'E - (AF - A'F') = CF' - CF$

$\Rightarrow AA' - GF = -F'F$

$\Rightarrow GF' = GF - F'F \quad$ ↯ $\Delta GF'F$



□

(b) $\Rightarrow$ (a) $\quad AE - AF = CE - CF \qquad (1)$

$ABCD$ - tangentni

II *varijanta*

$AG = AF \Rightarrow EG = AE - AF \qquad (2)$

$CH = CF \Rightarrow EH = CE - CF \qquad (3)$

$(2) \Rightarrow \Delta AGF$ - jednakokraki

$\Rightarrow s_{GF} \equiv$ sim. $\angle BAD \qquad (4)$

$(3) \Rightarrow \Delta HFC$ - jednakokraki

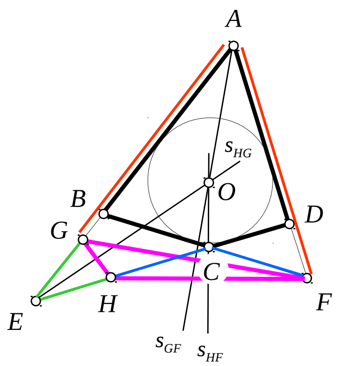
$\Rightarrow s_{HF} \equiv$ sim. $\angle HCF \equiv$ sim. $\angle BCD \qquad (5)$

$(1), (2), (3) \Rightarrow \Delta EHG$ - jednakokraki

$\Rightarrow s_{HG} \equiv$ sim. $\angle HEG \qquad (6)$

$s_{GF} \cap s_{HF} \cap s_{HG} = \{O\} \qquad (7)$

O - centar op. kružn. za ΔHFG



$(4), (5), (6), (7) \Rightarrow$

O - centar upisane kružn. u $ABCD$

$\Rightarrow ABCD$ - tangentni

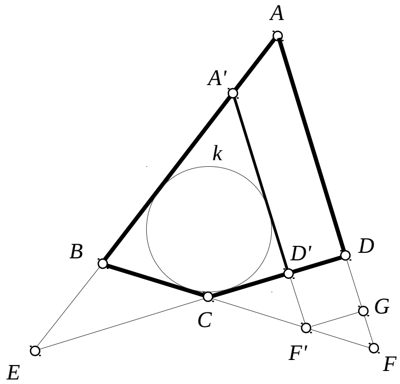
(c) ⇒ (a) $BE + BF = DE + DF$ (1)

ABCD - tangentni

I *varijanta*

k - dodiruje AB, BC, CD

1° k dodiruje DA ⇒ $ABCD$ - tangentni ✓



2° k ne dodiruje DA

$A'D'$ - tangenta k i $A'D' \parallel AD$

$A'D' \cap CF = \{F'\}$ $F'G \parallel CD$

$A'BCD'$ - tangentni

((a) ⇒ (c)) ⇒ $BE + BF' = D'E + D'F'$ (2)

(1) - (2) ⇒ $BF - BF' = (DE - D'E) + (DF - D'F')$

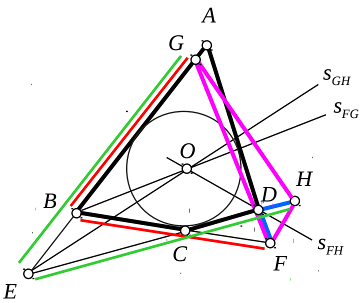
⇒ $F'F = DD' + (DF - DG)$

⇒ $F'F = F'G + GF$ ▯ $\Delta F'FG$

(c) ⇒ (a) $BE + BF = DE + DF$ (1)

ABCD - tangentni

II *varijanta*



$BG = BF$ ⇒ $EG = BE + BF$ (2)

$DH = DF$ ⇒ $EH = DE + DF$ (3)

(2) ⇒ ΔBFG - jednakokraki

⇒ $s_{FG} \equiv$ sim. ▯ABC (4)

(3) ⇒ ΔDFH - jednakokraki

⇒ $s_{FH} \equiv$ sim. ▯$HDF \equiv$ sim. ▯CDA (5)

(1), (2), (3) ⇒ ΔEHG - jednakokraki

⇒ $s_{GH} \equiv$ sim. ▯HEG (6)

$s_{FG} \cap s_{FH} \cap s_{GH} = \{O\}$ (7)

O - centar op. kružn. za ΔHFG

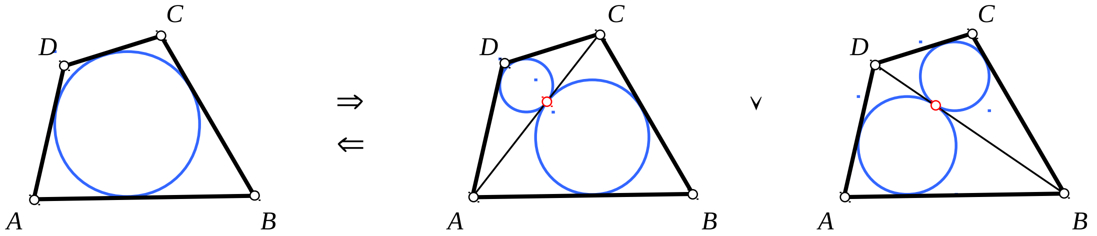
(4), (5), (6), (7) ⇒

O - centar upisane kružn. u $ABCD$

⇒ $ABCD$ - tangentni

■

Primer 7. Dokazati da je četvorougao *ABCD* tangentan ako i samo ako se kružnice upisane u trouglove *ABC* i *ADC* dodiruju, odnosno kružnice upisane u trouglove *ABD* i *BCD* dodiruju.



Rešenje. (⇒) *ABCD* - tangentni

T 3 ⇒ $AB + CD = AD + BC$ (1)

k_D - upisana u ΔABC k_B - upisana u ΔADC

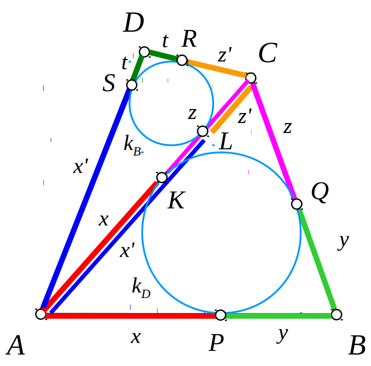
pretp. k_D i k_B se ne dodiruju ⇒ $k_D \cap AC = \{K\}$ $k_D \cap AC = \{L\}$ $K \neq L$

$k_D \cap AB = \{P\}$ $k_D \cap BC = \{Q\}$ $k_B \cap CD = \{R\}$ $k_B \cap AD = \{S\}$

$AP = AK = x$ $AS = AL = x' > x$ (2)

$CQ = CK = z$ $CR = CL = z' < z$ (3)

$BP = BQ = y$ $DR = DS = t$



$AB + CD = (x + y) + (z' + t) = x + y + z' + t$

$AD + BC = (x' + t) + (y + z) = x' + y + z + t$

(2), (3) ⇒ $AB + CD < AD + BC$ ↯ (1)

Slično za k_C i k_A

(⇐) $k_D \cap k_B \cap AC = \{K\}$

$k_D \cap AB = \{P\}$ $k_D \cap BC = \{Q\}$

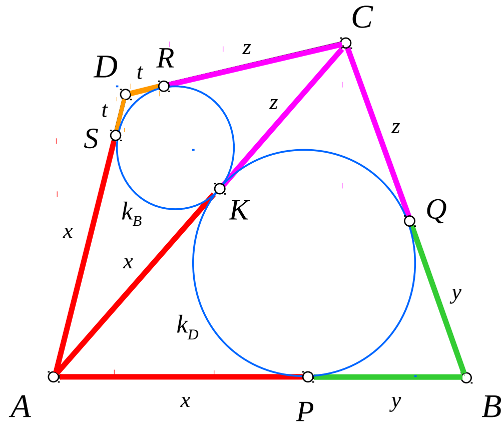
$k_B \cap CD = \{R\}$ $k_B \cap AD = \{S\}$

$AP = AK = AS = x$

$BP = BQ = y$

$CQ = CK = CR = z$

$DR = DS = t$



$AB + CD = (x + y) + (z + t) = x + y + z + t$

$AD + BC = (x + t) + (y + z) = x + y + z + t$

$\Rightarrow AB + CD = AD + BC \overset{\text{T 5.19}}{\Rightarrow} ABCD$ - tangentni

slično $k_A \cap k_C \cap BD = \{L\} \Rightarrow ABCD$ - tangentni

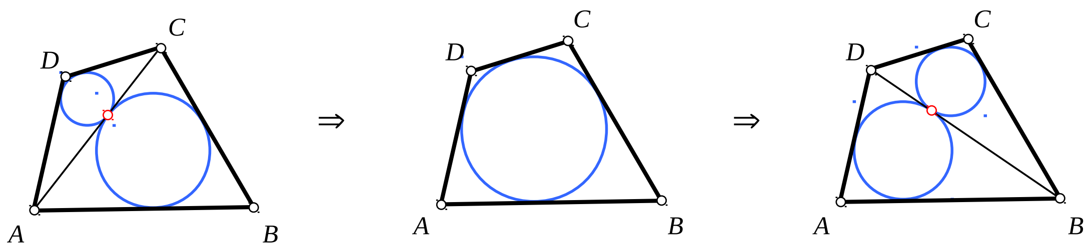
■

Primer 8. U četvorouglu *ABCD* kružnice upisane u trouglove *ABC* i *CDA* se dodiruju. Dokazati da se i kružnice upisane u trouglove *ABD* i *BCD* takođe dodiruju.

Rešenje.

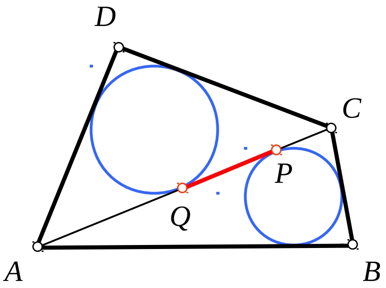
$k(A, B, C)$ i $k(C, D, A)$ se dodiruju $\overset{\text{Pr. 7 }(\Leftarrow)}{\Rightarrow} ABCD$ - tangentni

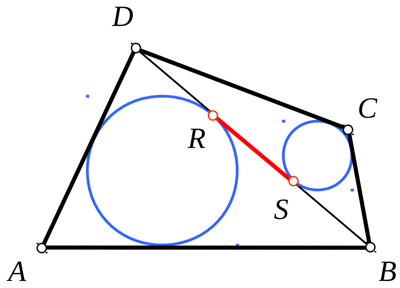
$\overset{\text{Pr. 7 }(\Rightarrow)}{\Rightarrow} k(A, B, D)$ i $k(B, C, D)$ se dodiruju



■

Primer 9. U konveksnom četvorouglu *ABCD* kružnice upisane u trouglove *ABC* i *CDA* dodiruju dijagonalu *AC* u tačkama *P* i *Q*, a kružnice upisane u trouglove *ABD* i *BCD* dijagonalu *BD* u tačkama *R* i *S*. Dokazati da je $PQ = RS$.





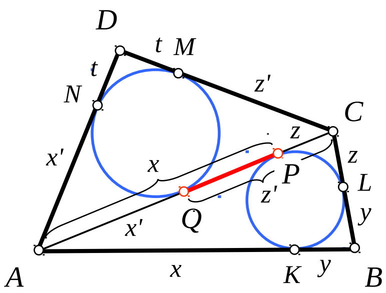
Rešenje. $AK = AP = x \quad AN = AQ = x' \qquad x \geq x'$

$CL = CP = z \quad CM = CQ = z' \qquad z' \geq z$

$BK = BL = y \quad DM = DN = t$

$PQ = AP - AQ = x - x' \qquad (1)$

$PQ = CQ - CP = z' - z \qquad (2)$



$$(1), (2) \Rightarrow 2PQ = x - x' + z' - z$$
$$= (x + z') - (x' + z)$$
$$= (x + \mathbf{y} + z' + \mathbf{t}) - (x' + \mathbf{t} + \mathbf{y} + z)$$
$$= (AB + CD) - (AD + BC)$$

$PQ = \frac{1}{2}((AB + CD) - (AD + BC))$ za $x \geq x'$

$PQ = \frac{1}{2}((AD + BC) - (AB + CD))$ za $x < x'$

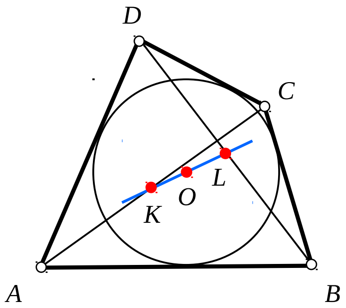
$\Rightarrow PQ = \frac{1}{2}|(AD + BC) - (AB + CD)|$

slično $RS = \frac{1}{2}|(AB + CD) - (AD + BC)|$

$\Rightarrow PQ = RS$

■

Primer 10. (Njutn) Centar upisane kružnice u tangentan četvorougao leži na pravoj koja spaja sredine dijagonala. Dokazati



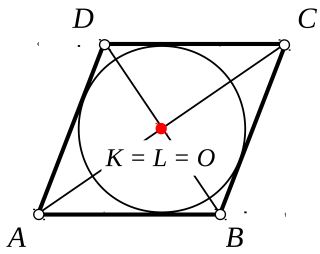
Rešenje. $ABCD$ - tangentan

K, L - središta AC i BD

O - centar upisane kružnice

(a) $ABCD$ - paralelogram (romb)

$K = L = O$ ✓

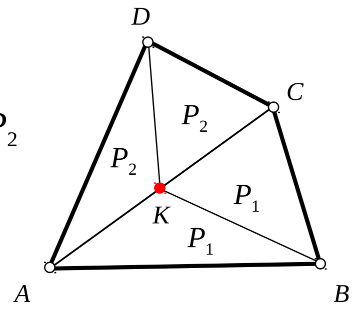


(b) $ABCD$ nije pralelogram $AB \nparallel CD$

$P_{ABCD} = P$

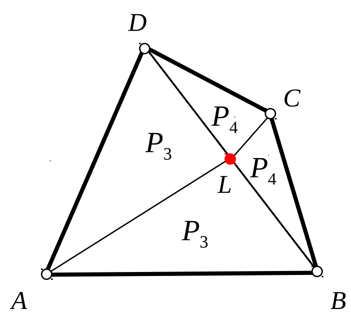
K - sred. $AC \Rightarrow P_{ABK} = P_{BCK} = P_1 \wedge P_{ADK} = P_{CDK} = P_2$

$$P_{ABK} + P_{CDK} = P_1 + P_2 = \frac{1}{2}P \qquad (1)$$

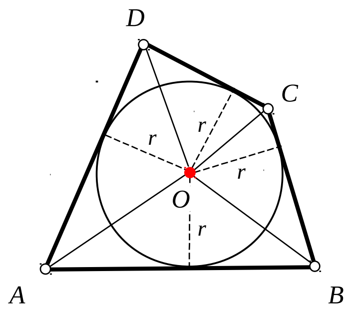


L - sred. $BD \Rightarrow P_{ABL} = P_{ADL} = P_3 \wedge P_{BCL} = P_{CDL} = P_4$

$$P_{ABL} + P_{CDL} = P_3 + P_4 = \frac{1}{2}P \qquad (2)$$



$ABCD$ - tangentan $\overset{T\ 3}{\Rightarrow}$ $AB + CD = AD + BC$ (3)



$$P_{ABO} + P_{CDO} = \frac{1}{2} AB \cdot r + \frac{1}{2} CD \cdot r$$
$$= \frac{1}{2}(AB + CD) \cdot r$$
$$\overset{(3)}{=} \frac{1}{2}(AD + BC) \cdot r$$
$$= \frac{1}{2} AD \cdot r + \frac{1}{2} BC \cdot r$$
$$= P_{ADO} + P_{BCO}$$

$P_{ABO} + P_{CDO} = P_{ADO} + P_{BCO}$ (4)

$P_{ABO} + P_{CDO} + P_{ADO} + P_{BCO} = P_{ABCD} = P$ (5)

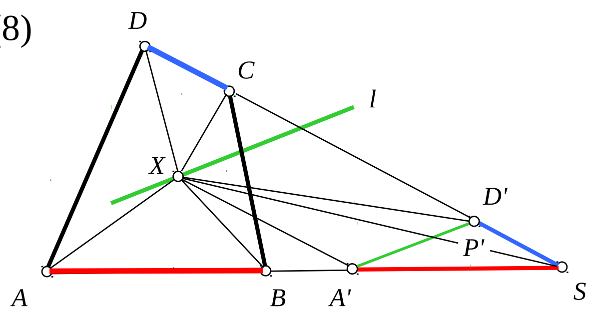
(4), (5) $\Rightarrow$ $P_{ABO} + P_{CDO} = \frac{1}{2} P$ (6)

(1), (2), (6) $\Rightarrow$ $P_{ABK} + P_{CDK} = P_{ABL} + P_{CDL} = P_{ABO} + P_{CDO} = \frac{1}{2} P$ (7)

$ABCD$ - proizvoljan četvorougao , $AB \nparallel CD$, $AB \cap CD = \{S\}$

$P_{ABCD} = P$

$X \in \text{int}\, ABCD$ $P_{ABX} + P_{CDX} = \frac{1}{2} P$ (8)



$A'S = AB$, $D'S = DC$

$P_{ABX} = P_{A'SX}$ $P_{CDX} = P_{SD'X}$

(8) $\Rightarrow$ $P_{A'SX} + P_{SD'X} = \frac{1}{2} P$

$P_{A'SX} + P_{SD'X} = P_{A'SD'X} = \frac{1}{2} P$ (9)

$P_{A'SD'X} = P_{A'SD'} + P_{A'D'X}$ $P_{A'SD'} = P' =$ const. (10)

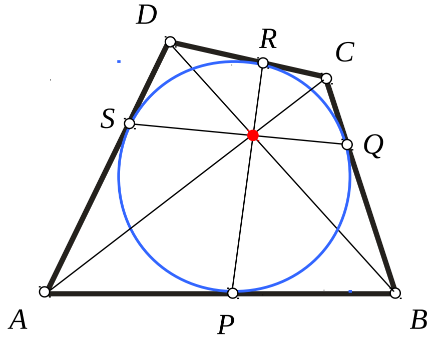
(9), (10) $\Rightarrow$ $P_{A'D'X} = P_{A'SD'X} - P_{A'SD'} = \frac{1}{2} P - P' =$ const.

$\Rightarrow$ $X \in l$, $l \parallel A'D'$ (11)

(7), (11) $\Rightarrow$ K, L, O - kolinearne

■

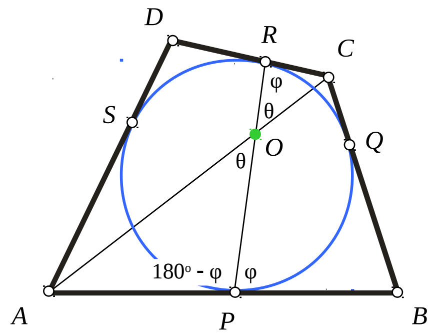
Primer 20. Neka je *ABCD* tangentni četvorougao i neka su *P*, *Q*, *R*, *S* tačke u kojima upisana kružnica dodiruje stranice *AB*, *BC*, *CD*, *DA*, redom. Dokazati da se dijagonale *AC* i *BD* i duži *PR* i *QS* seku u jednoj tački.



Rešenje. $AC \cap PR = \{O\}$

$\angle AOP = \angle COR = \theta$

$\angle BPO = \angle CRO = \varphi \quad \Rightarrow \quad \angle APO = 180^{o} - \varphi$

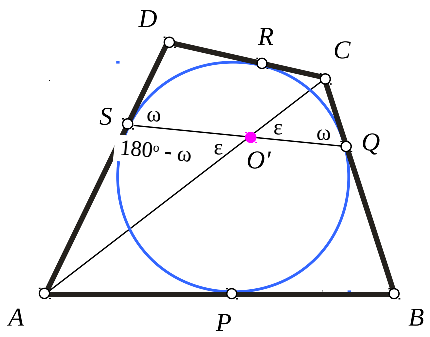


$$\frac{P_{APO}}{P_{CRO}} = \frac{\frac{1}{2} AO \cdot OP \cdot \sin\theta}{\frac{1}{2} CO \cdot OR \cdot \sin\theta} = \frac{AO \cdot OP}{CO \cdot OR} \qquad (1)$$

$$\frac{P_{APO}}{P_{CRO}} = \frac{\frac{1}{2} AP \cdot OP \cdot \sin(180^{0} - \varphi)}{\frac{1}{2} CR \cdot OR \cdot \sin\varphi} = \frac{\frac{1}{2} AP \cdot OP \cdot \sin\varphi}{\frac{1}{2} CR \cdot OR \cdot \sin\varphi} = \frac{AP \cdot OP}{CR \cdot OR} \qquad (2)$$

$$(1), (2) \Rightarrow \quad \frac{AO \cdot \cancel{OP}}{CO \cdot \cancel{OR}} = \frac{AP \cdot \cancel{OP}}{CR \cdot \cancel{OR}} \Rightarrow \frac{AO}{CO} = \frac{AP}{CR} \qquad (3)$$

$$\frac{AO}{CO} = \frac{AP}{CR} \qquad (3)$$



$$AC \cap QS = \{O'\}$$

$$\angle AO'S = \angle CO'Q = \varepsilon$$

$$\angle DSO' = \angle CQO' = \omega \quad \Rightarrow \quad \angle ASO' = 180^{\circ} - \omega$$

$$\frac{P_{ASO'}}{P_{CQO'}} = \frac{\frac{1}{2} AO' \cdot O'S \cdot \sin\varepsilon}{\frac{1}{2} CO' \cdot O'Q \cdot \sin\varepsilon} = \frac{AO' \cdot O'S}{CO' \cdot O'Q} \qquad (4)$$

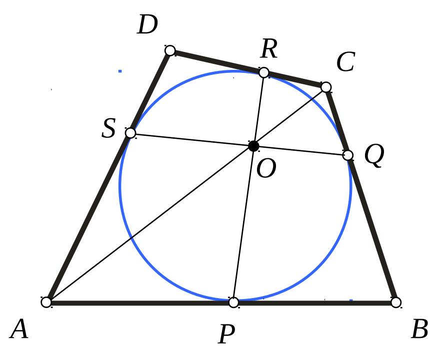
$$\frac{P_{ASO'}}{P_{CQO'}} = \frac{\frac{1}{2} AS \cdot O'S \cdot \sin(180^{0} - \omega)}{\frac{1}{2} CQ \cdot O'Q \cdot \sin\omega} = \frac{\frac{1}{2} AS \cdot O'S \cdot \sin\omega}{\frac{1}{2} CQ \cdot O'Q \cdot \sin\omega} = \frac{AS \cdot O'S}{CQ \cdot O'Q} \qquad (5)$$

$$(4), (5) \Rightarrow \frac{AO' \cdot \cancel{O'S}}{CO' \cdot \cancel{O'Q}} = \frac{AS \cdot \cancel{O'S}}{CQ \cdot \cancel{O'Q}} \Rightarrow \frac{AO'}{CO'} = \frac{AS}{CQ} \qquad (6)$$

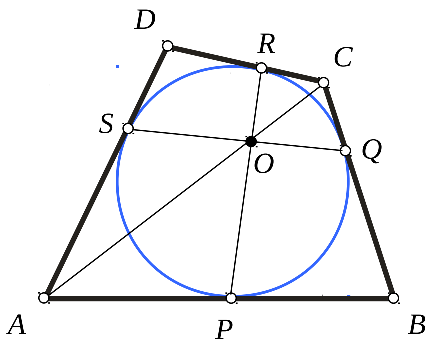
$$AS = AP \wedge CQ = CR \overset{(6)}{\Rightarrow} \frac{AO'}{CO'} = \frac{AP}{CR} \qquad (7)$$

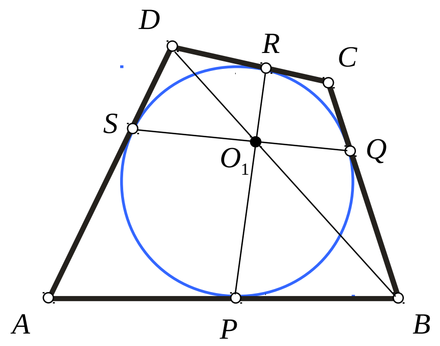
$$\left.\begin{aligned} \frac{AO}{CO} &= \frac{AP}{CR} \qquad (3) \\ \frac{AO'}{CO'} &= \frac{AP}{CR} \qquad (7) \end{aligned}\right\} \Rightarrow \frac{AO}{CO} = \frac{AO'}{CO'} \Rightarrow O \equiv O'$$

$$\Rightarrow AC \cap PR \cap QS = \{O\}$$



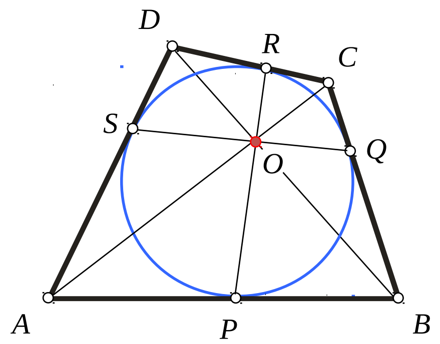
$AC \cap \underline{PR \cap QS} = \{O\}$ slično $BD \cap \underline{PR \cap QS} = \{O_1\}$





$O \in PR \cap QS \ \wedge\ O_1 \in PR \cap QS \ \Rightarrow\ O \equiv O_1$

$\Rightarrow\ AC \cap BD \cap PR \cap QS = \{O\}$



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