GEOMETRIJSKA MESTA TAČAKA

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GMT(α) – *geometrijsko mesto tačaka*

⇔

skup svih tačaka u zadatoj oblasti koje imaju zadatu osobinu α

osnovni problem: odrediti GMT(α).

Odrediti GM tačaka u ravni čija su rastojanja od dve date tačke A i B jednaka.

Odrediti GM tačaka u ravni čija su rastojanja od dve date prave a i b jednaka.

Odrediti GM tačaka u ravni iz kojih se data duž vidi pod datim uglom.

Date su tačke A i B. Odrediti GM tačaka C u ravni, takvih da je ΔABC:
(a) oštrougli; (b) pravougli; (c) tupougli.

Odrediti GM tačaka u unutrašnjosti datog ugla čija su rastojanja od krakova jednaka.

Odrediti GM tačaka X u unutrašnjosti datog jednakostraničnog ΔABC, takvih da od rastojanja X od pravih BC, CA, AB može da se konstruiše trougao.

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struktura rešenja

$\text{GMT}(\alpha) = ?$ G – pretpostavka (hipoteza) o GMT

zadatak: dokazati $\text{GMT}(\alpha) = \text{G}$

varijante dokaza

1. $\text{GMT}(\alpha) \subset \text{G} \;\wedge\; \text{G} \subset \text{GMT}(\alpha)$
   $X \in \text{GMT} \Rightarrow X \in \text{G} \;\wedge\; X \in \text{G} \Rightarrow X \in \text{GMT}$
2. $\text{GMT}(\alpha) \subset \text{G} \;\wedge\; \overline{\text{GMT}(\alpha)} \subset \overline{\text{G}}$
   $X \in \text{GMT} \Rightarrow X \in \text{G} \;\wedge\; X \notin \text{GMT} \Rightarrow X \notin \text{G}$
3. $\overline{\text{G}} \subset \overline{\text{GMT}(\alpha)} \;\wedge\; \text{G} \subset \text{GMT}(\alpha)$
   $X \notin \text{G} \Rightarrow X \notin \text{GMT} \;\wedge\; X \in \text{G} \Rightarrow X \in \text{GMT}$
4. $\overline{\text{G}} \subset \overline{\text{GMT}(\alpha)} \;\wedge\; \overline{\text{GMT}(\alpha)} \subset \overline{\text{G}}$
   $X \notin \text{G} \Rightarrow X \notin \text{GMT} \;\wedge\; X \notin \text{GMT} \Rightarrow X \notin \text{G}$

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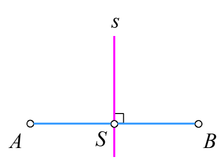
Primer 1. *Date su tačke A i B. Odrediti GM tačaka X, takvih da je XA = XB.*



Rešenje. $\text{G} = s$ – simetrala AB

1. $X \in \text{GMT} \Rightarrow X \in s$

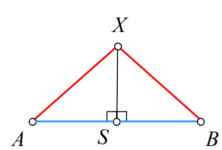
   $XA = XB$

   $X \equiv S \Rightarrow X \in s$

   $X \neq S$

   $\Delta XAS \cong \Delta XBS$ (SSS)

   $\Rightarrow \angle XSA = \angle XSB = 90^0 \Rightarrow X \in s$



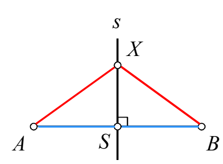
$X \in s \Rightarrow X \in \text{GMT}$

$X \equiv S \Rightarrow X \in \text{GMT}$

$X \neq S$

$\Delta XAS \cong \Delta XBS$ (SUS)

$\Rightarrow XA = XB \Rightarrow X \in \text{GMT}$



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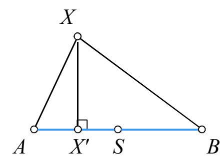
2. $X \in \text{GMT} \Rightarrow X \in s$ kao 1.



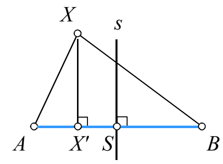
$X \notin \text{GMT} \Rightarrow X \notin s$

$\underline{XA < XB} \vee XA > XB$

$XX' \perp AB$

$AX' = \sqrt{AX^2 - XX'^2} < \sqrt{BX^2 - XX'^2} = BX' \Rightarrow X' \neq S \Rightarrow X \notin s$

3. $X \notin s \Rightarrow X \notin \text{GMT}$



$XX' \perp AB \Rightarrow X' \neq S$

$\Rightarrow \underline{X'A < X'B} \vee X'A > X'B$

$XA = \sqrt{X'A^2 + XX'^2} < \sqrt{X'B^2 + XX'^2} = XB \Rightarrow X \notin \text{GMT}$

$X \in s \Rightarrow X \in \text{GMT}$ kao 1.

4. $X \notin s \Rightarrow X \notin \text{GMT}$ kao 3.

$X \notin \text{GMT} \Rightarrow X \notin s$ kao 2.

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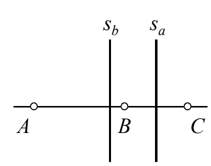
Primer 2. *Date su tri različite tačke A, B, C. Odrediti GM tačaka X, takvih da je XA = XB = XC.*

Rešenje. s_a – sim. BC s_b – sim. CA

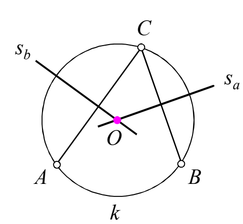
Pr. 1 $\Rightarrow X \in s_a \cap s_b$

(a) A, B, C – kolinearne



$s_a \parallel s_b \Rightarrow s_a \cap s_b = \varnothing \Rightarrow \text{GMT} = \varnothing$

(b) A, B, C – nekolinearne



$s_a \cap s_b = \{O\} \Rightarrow \text{GMT} = O$

O – centar kružnice $k(A, B, C)$

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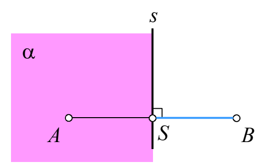
Primer 3. *Date su tačke A i B. Odrediti GM tačaka X, takvih da je XA < XB.*

Rešenje. s – simetrala AB

$G = \alpha\ (s, A)$ – otv. poluravan; ivica s, sadrži A

G = GMT

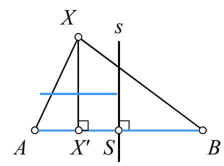


(a) $X \in G = \alpha \qquad XX' \perp AB,\ X' \in AB$

$\Rightarrow X' \in pp(S, A) \Rightarrow X'A < X'B$

$\Rightarrow XA = \sqrt{X'A^2 + XX'^2} < \sqrt{X'B^2 + XX'^2} = XB$

$\Rightarrow XA < XB \Rightarrow X \in \text{GMT}$



(b) $X \in \text{GMT}$

$\Rightarrow XA < XB \Rightarrow X'A < X'B \Rightarrow X' \in pp(S, A)$

$\Rightarrow X' \in \alpha \Rightarrow X \in \alpha = G$

(a), (b) $\Rightarrow$ GMT $= \alpha$

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Primer 4. *Date su tačke A, B, C. Odrediti GM tačaka X u ravni ABC, takvih da je XA < XB < XC.*

Rešenje. s_c – sim. AB $\quad s_a$ – sim. BC

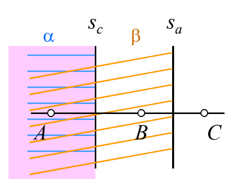
(a) A, B, C – kolinearne

1° $A - B - C$

$XA < XB \overset{\text{Pr. 3}}{\Rightarrow} X \in \alpha \qquad XB < XC \overset{\text{Pr. 3}}{\Rightarrow} X \in \beta$

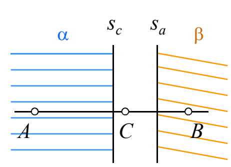
$XA < XB < XC \Rightarrow X \in \alpha \cap \beta = \alpha$

$\Rightarrow$ GMT $= \alpha$



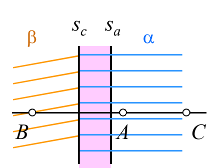
2° $A - C - B$

GMT $= \alpha \cap \beta = \varnothing$



3° $B - A - C$

GMT $= \alpha \cap \beta =$ "pruga" između s_a i s_c



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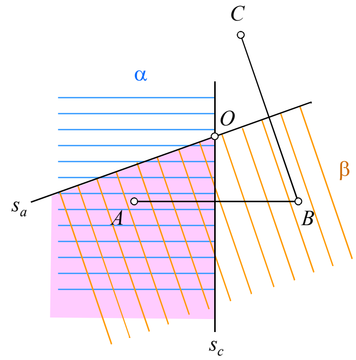
(b) A, B, C – nekolinearne

$s_a \cap s_c = \{O\}$

$XA < XB \Rightarrow X \in \alpha \quad XB < XC \Rightarrow X \in \beta$

$XA < XB < XC \Rightarrow X \in \alpha \cap \beta = \text{int} \angle s_a O s_c$

$\Rightarrow \text{GMT} = \text{int} \angle s_a O s_c$



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Primer 5. *Dat je ugao aOb. Odrediti GM tačaka X u unutrašnjosti ugla aOb čija su rastojanja od krakova a i b međusobno jednaka.*

Rešenje. (a) $X \in \text{GMT}$

A, B – norm. projekcije X na $p(a), p(b)$

$XA = XB$

$\Rightarrow A \in a, B \in b$ – objašnjenje (∗) sledeći slajd

$\Delta XOA \cong \Delta XOB$ (SSU)

$\Rightarrow \angle XOA \cong \angle XOB = \varphi$

$\Rightarrow X \in s$ – simetrala $\angle aOb$

$\text{G} = s$

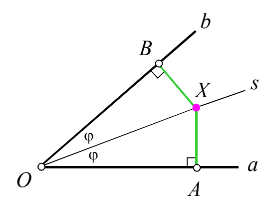
(b) $X \in s \quad \angle XOA \cong \angle XOB = \varphi$

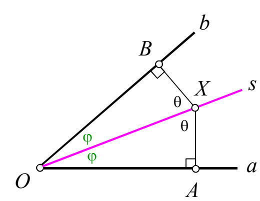
$\Rightarrow \angle OXA \cong \angle OXB = 90^0 - \varphi = \theta$

$\Rightarrow \Delta XOA \cong \Delta XOB$ (USU) $\Rightarrow XA = XB$

$\Rightarrow X \in \text{GMT}$

(a) , (b) $\Rightarrow$ GMT = s





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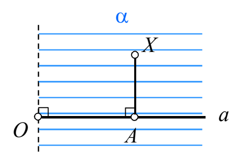
(∗) $X \in \text{int}\,\angle aOb \wedge X \in \text{GMT} \;\Rightarrow\; A \in a \wedge B \in b$

$A \in a \;\Leftrightarrow\; X \in \alpha$

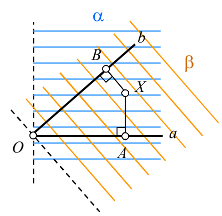


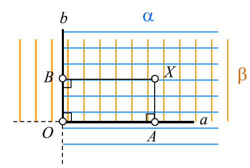
$\angle aOb \leq 90^0$

$\alpha \cap \beta = \text{int}\,\angle aOb$

$\forall X \in \text{int}\,\angle aOb$

$\Rightarrow\; A \in a \wedge B \in b$





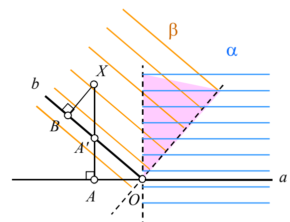
$\angle aOb > 90^0$

$\alpha \cap \beta \subset \text{int}\,\angle aOb$

pretp. $X \in \text{GMT} \;\wedge\; X \in \text{int}\,\angle aOb \setminus (\alpha \cap \beta)$

$\Rightarrow\; XA > XA' > XB$ ↯ $X \in \text{GMT} \;\Leftrightarrow\; XA = XB$

$\Rightarrow\; X \in \alpha \cap \beta \;\Rightarrow\; A \in a \wedge B \in b$



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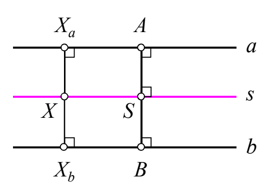
Primer 6. *Date su prave a i b. Odrediti GM tačaka X čija su rastojanja od a i b međusobno jednaka.*

Rešenje.

(a) $a \parallel b \quad A \in a\,,\, B \in b \quad AB \perp a, b$

s – simetrala AB

$G = s$



1° $X \in s \quad XX_a \perp a \quad XX_b \perp b$

$XX_a = SA = SB = XX_b \Rightarrow XX_a = XX_b$

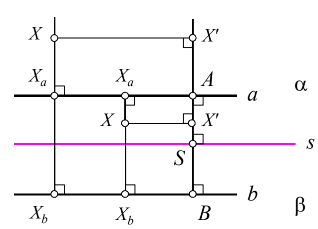
$\Rightarrow\; X \in \text{GMT}$

2° $X \notin s \quad X \in \underline{\alpha(a)} \cup \beta(b)$

$XX_a = X'A < X'B = XX_b \Rightarrow XX_a < XX_b$

$\Rightarrow\; X \notin \text{GMT}$

1°, 2° $\Rightarrow$ GMT $= s$



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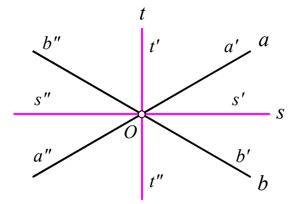
(b) $a \cap b = \{O\}$

$X \in \angle a'Ob' \overset{\text{Pr. 5}}{\Rightarrow} \text{GMT} = s'$

$X \in \angle a''Ob'' \overset{\text{Pr. 5}}{\Rightarrow} \text{GMT} = s''$

$X \in \angle a'Ob'' \overset{\text{Pr. 5}}{\Rightarrow} \text{GMT} = t'$

$X \in \angle a''Ob' \overset{\text{Pr. 5}}{\Rightarrow} \text{GMT} = t''$



$\Rightarrow \text{GMT} = (s' \cup s'') \cup (t' \cup t'') = s \cup t$

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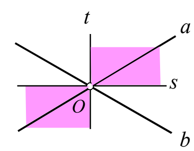
Primer 7. *Date su prave a i b. Odrediti GM tačaka X, takvih da je $d(X, a) < d(X, b)$. ($d(X, a)$ – rastojanje tačkeX od prave a.)*

Rešenje.

$a \parallel b$

$a \cap b = \{O\}$



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Primer 8. *Dat je ugao aOb. Odrediti GM tačaka X u ravni ugla aOb čija su rastojanja od krakova a i b međusobno jednaka.*

Rešenje. $d(X, F)$ – rastojanje tačke X od figure F $\quad d(X, F) = \min_{Y \in F} XY$

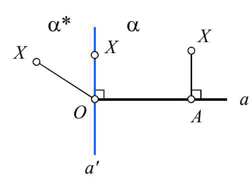
$a(O)$ – zatvorena poluprava; početak O $\quad a'(O) \perp a$

$\alpha(a)$ – otvorena poluravan; ivica a'

α^* – komplementarna poluravan za α

$X \in \alpha \Rightarrow d(X, a) = XA$

$X \in \alpha^* \cup a' \Rightarrow d(X, a) = XO$



$b'(O) \perp b$

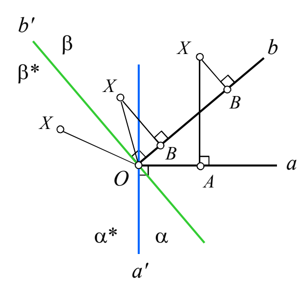
$X \in \beta \Rightarrow d(X, b) = XB$

$X \in \beta^* \cup b' \Rightarrow d(X, b) = XO$

$X \in \alpha \cap \beta \Rightarrow d(X, a) = XA,\ d(X, b) = XB$

$X \in \alpha^* \cap \beta^* \Rightarrow d(X, a) = d(X, b) = XO$

$X \in \alpha^* \cap \beta \Rightarrow d(X, a) = XO,\ d(X, b) = XB$



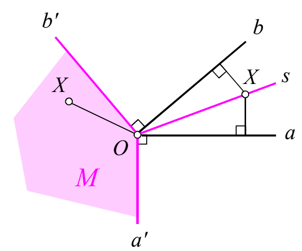
$G = s \cup M \quad M = \angle a'Ob' \cup \text{int}\, \angle a'Ob'$

(a) $X \in G$

$X \in s \overset{\text{Pr. 5}}{\Rightarrow} d(X, a) = d(X, b)$

$X \in M \Rightarrow d(X, a) = XO = d(X, b)$

$\Rightarrow X \in \text{GMT}$



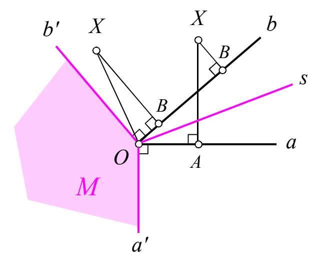
(b) $X \notin G$

$\underline{X \in \text{int}\, \angle sOb'} \cup \text{int}\, \angle sOa'$

$d(X, a) = XO > XB = d(X, b)$

$d(X, a) = XA > XB = d(X, b)$

$\Rightarrow X \notin \text{GMT}$



(a), (b) $\Rightarrow$ GMT $= s \cup M$

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Primer 9. *Dati su duž AB i ugao φ. Odrediti GM tačaka X, takvih da je $\angle AXB = \varphi$.*

Rešenje. $X \in \text{GMT} \quad X \in \alpha$

$k(A, B, X) \qquad l(A, X, B) \subset \alpha$

$G = l$ u α

(a) $Y \in G$

$\angle AYB = \angle AXB = \varphi \Rightarrow Y \in \text{GMT}$

(b) $Y \notin G$

$Y \in \text{int}\, k$

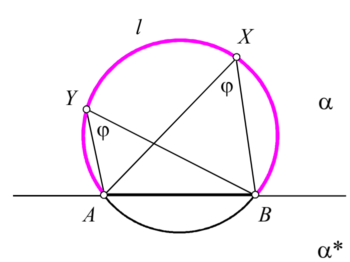
$AY \cap l = \{A, Y_1\}$

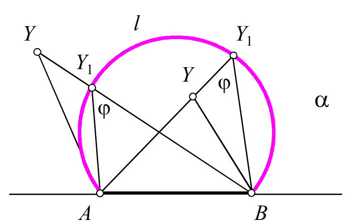
$\angle AYB > \angle AY_1B = \varphi \Rightarrow Y \notin \text{GMT}$

$Y \in \text{int}\, k$

$AY \cap l = \{A, Y_1\} \vee \underline{BY \cap l = \{B, Y_1\}}$

$\angle AYB < \angle AY_1B = \varphi \Rightarrow Y \notin \text{GMT}$





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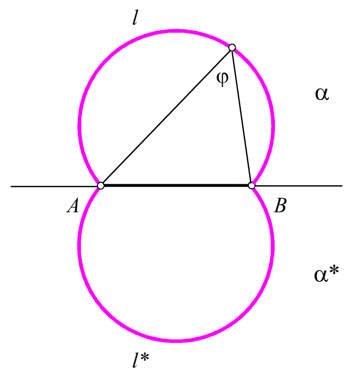
(a), (b) $\Rightarrow$ GMT (u α) = luk l (bez A i B)

slično GMT (u α^*) = luk l^* (bez A i B)

$\Rightarrow$ GMT = $l \cup l^*$



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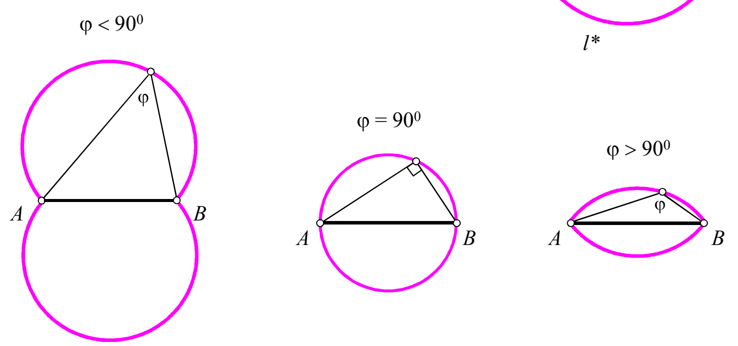

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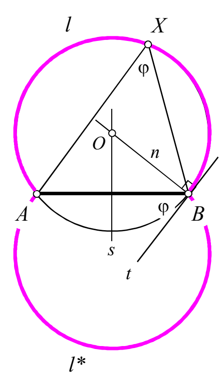
Primer 10. *Konstruisati GMT iz primera 9, tj. konstruisati GMT iz kojih se data duž AB vidi pod datim uglom φ.*

Rešenje.

Kontrukcija

1. $t(B)$, $\angle ABt = \varphi$
2. $n(B)$, $n \perp t$
3. s – simetrala AB
4. $s \cap n = \{O\}$
5. $k(O; r = OA = OB)$
6. l – luk AB na k bez A i B (l i φ s raznih strana AB)
7. GMT = $l \cup l^*$, $l^* = \sigma_{AB}(l)$



Dokaz

Teorema o uglu između tangente i tetive kružnice

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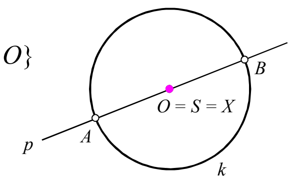
Primer 11. *Dati su tačka S i kružnica k(O). Neka je p proizvoljna prava kroz S koja seče k u tačkama A i B. Odrediti GM tačaka X – sredina duži AB.*

Rešenje. (a) $S \equiv O$

$\Rightarrow$ AB prečnik k $\Rightarrow$ $X = O$ za $\forall\, p(S)$ $\Rightarrow$ GMT = $\{O\}$



(b) $S \neq O$

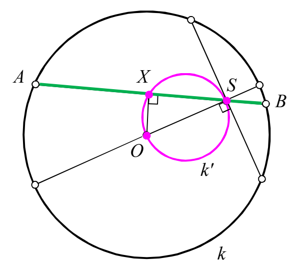
$X \in$ GMT , $X \neq O, S$ $\Rightarrow$ $\angle OXA = \angle OXB = \angle OXS = 90^0$

Pr. 9 $\Rightarrow$ $X \in k'$ – prečnik OS

(b1) $S \in \text{int}\, k$

GMT = k'

$O \in$ GMT $\quad$ $S \in$ GMT



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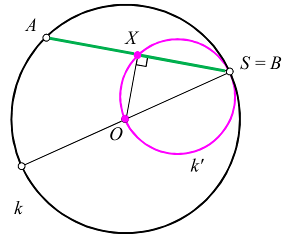
(b2) $S \in k$

GMT = $k' - \{S\}$

$O \in$ GMT $\quad$ $S \notin$ GMT

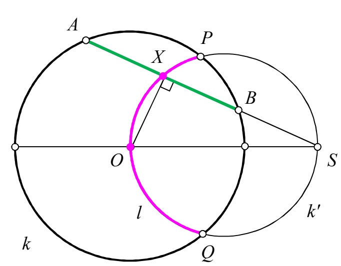


(b3) $S \in \text{ext}\, k$

GMT = $k' \cap \text{int}\, k$

$= l$ – otvoren luk PQ

$O \in$ GMT



■

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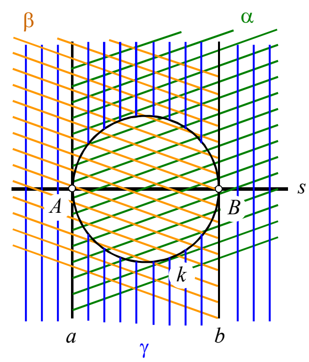
Primer 12. *Date su tačke A i B. Odrediti GM tačaka X , takvih da je ΔABX:* (a) *oštrougli*; (*b*) *pravougli*; (*c*) *tupougli.*

Rešenje. $p(A, B) = s$

$a(A) \perp s$ $b(B) \perp s$ $k(AB)$ – prečnik AB

$\alpha = \text{pr}(a, B)$ $\beta = \text{pr}(b, A)$ $\gamma = \text{ext}\, k$



$X \in$GMT

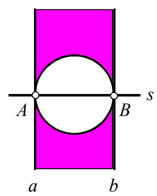
(a) $\angle A < 90^0 \Rightarrow X \in \alpha - s$

$\angle B < 90^0 \Rightarrow X \in \beta - s$

$\angle C < 90^0 \Rightarrow X \in \gamma - s$

$X \in (\alpha - s) \cap (\beta - s) \cap (\gamma - s)$

GMT $= (\alpha - s) \cap (\beta - s) \cap (\gamma - s)$



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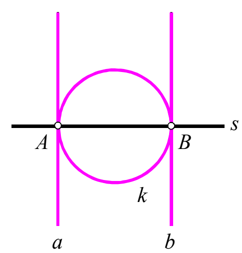
(b) $\angle A = 90^0 \Rightarrow X \in a - \{A\}$

$\angle B = 90^0 \Rightarrow X \in b - \{B\}$

$\angle C = 90^0 \Rightarrow X \in k - \{A, B\}$

GMT $= a \cup b \cup k - \{A, B\}$

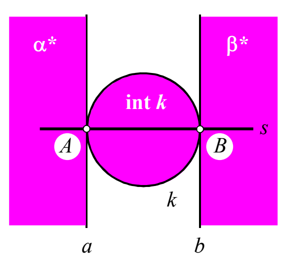


(c) $\angle A > 90^0 \Rightarrow X \in \alpha^*$

$\angle B > 90^0 \Rightarrow X \in \beta^*$

$\angle C > 90^0 \Rightarrow X \in \text{int}\, k$

GMT $= \alpha^* \cup \beta^* \cup \text{int}\, k - s$



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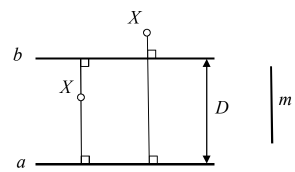
Primer 13. *Date su prave a i b i duž m. Odrediti GM tačaka X čiji zbir rastojanja od pravih a i b jednak m.*

Rešenje.

(a) $a \parallel b \quad D = d(a, b)$ – rastojanje između a i b

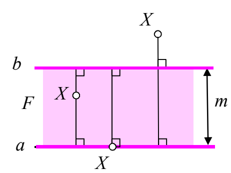


1° $m < D$

$d(X, a) + d(X, b) \geq D > m \quad \forall X$

$\Rightarrow$ GMT = $\varnothing$

2° $m = D$



$X \in F \cup \{a, b\} \Rightarrow d(X, a) + d(X, b) = m$

$\Rightarrow X \in$ GMT

G = $F \cup \{a, b\}$

$X \notin$ G $\Rightarrow d(X, a) + d(X, b) > m$

$\Rightarrow X \notin$ GMT

GMT = $F \cup \{a, b\}$

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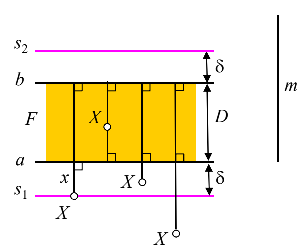
3° $m > D$



$X \in F \cup \{a, b\}$

$d(X, a) + d(X, b) = d(a, b) = D < m$

$\Rightarrow X \notin$ GMT

$X \notin F \cup \{a, b\} \qquad d(X, a) = x$

$d(X, a) + d(X, b) = x + (x + D) = 2x + D$

$X \in$ GMT $\Rightarrow 2x + D = m \Rightarrow x = \frac{m - D}{2}$

$s_1, s_2 \parallel a, b \quad d(s_1, a) = d(s_2, b) = \delta = \frac{m - D}{2}$

G = $s_1 \cup s_2$

$X \in$ G $\Rightarrow d(X, a) + d(X, b) = 2x + D = 2\frac{m - D}{2} + D = m \Rightarrow X \in$ GMT

$X \notin$ G $\Rightarrow d(X, a) + d(X, b) < m \vee d(X, a) + d(X, b) > m \Rightarrow X \notin$ GMT

$\Rightarrow$ GMT = $s_1 \cup s_2$

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(b) $a \cap b = \{O\}$

Ako je GMT $\neq \emptyset$, kako naći jednu tačku $X \in$GMT?

$X \in a \Rightarrow d(X, a) = 0 \Rightarrow d(X, b) = m$

$b' \parallel b$, $d(b', b) = m$ $b' \cap a = \{A\}$

$d(A, a) + d(A, b) = 0 + m = m \Rightarrow A \in$GMT

$b'' \parallel b$, $a' \parallel a$, $a'' \parallel a$

$d(b'', b) = d(a', a) = d(a'', a) = m$

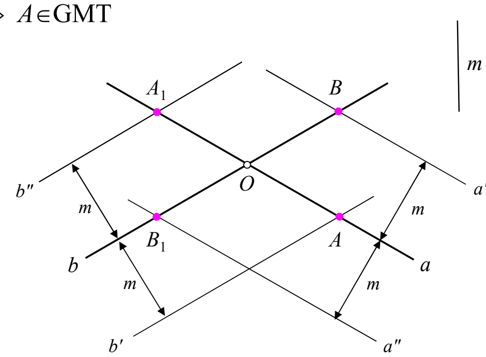
$b'' \cap a = \{A_1\}$

$a' \cap b = \{B\}$

$a'' \cap b = \{B_1\}$

$A_1, B, B_1 \in$GMT

GMT = $\{A, B, A_1, B_1\}$?



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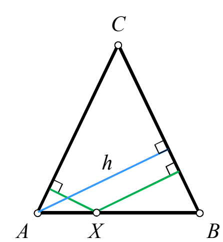
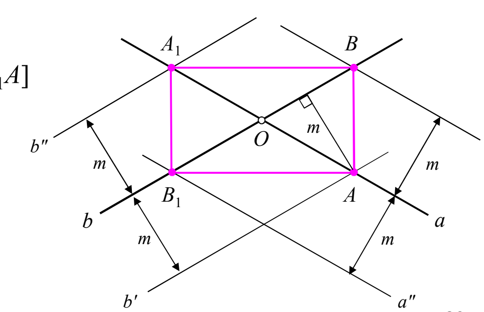
?!

Ne.

Lema 1. *Neka ABC jednakokraki trougao sa osnovicom AB i visinom na krak h. Tada za svaku tačku X osnovice AB važi* $d(X, BC) + d(X, AC) = h$. ■



G = $[AB] \cup [BA_1] \cup [A_1B_1] \cup [B_1A]$
= pravougaonik ABB_1A_1



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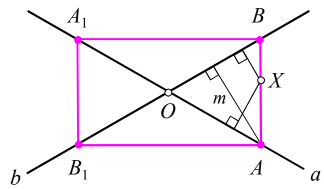
1^o $X \in G = ABCD$

$X \in [AB]$

$d(X, a) + d(X, b) \overset{(L)}{=} d(A, b) = m$

$\Rightarrow$ $X \in$GMT



2^o $X \notin G = ABCD$

$X \in \underline{\text{ext } ABCD} \cup \text{int } ABCD$

(1) $X \in \angle AOB \cup \text{int } \angle AOB$

$s(X) \parallel AB$, $s \cap a = \{A'\}$, $s \cap b = \{B'\}$

$d(X, a) + d(X, b) \overset{(L)}{=} d(A', b) > d(A, b) = m$

$\Rightarrow$ $X \notin$GMT

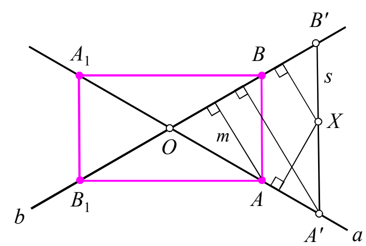
slično za $\angle BOC$, $\angle COD$, $\angle DOA$

(2) $X \in \text{int } ABCD$

. . .

$d(X, a) + d(X, b) < m$

$\Rightarrow$ $X \notin$GMT



1^o, 2^o $\Rightarrow$ GMT $= ABA_1B_1$ ■

Primer 14. *U ravni α date su tačke A i B i duži p i q. Odrediti u GM tačaka X u ravni α, takvih da je $XA : XB = p : q$.*

Rešenje. (a) $p = q$ GMT = simetrala AB (primer 1)

(b) $p \neq q$, $p > q$ $s = p(A, B)$

pretraga ravni α

$X \in$GMT

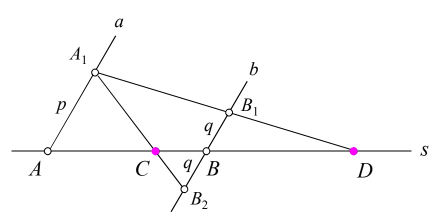
1^o $X \in s$

$a(A) \parallel b(B)$

$A_1 \in a$, $AA_1 = p$

$B_1, B_2 \in b$, $BB_1 = BB_2 = q$

$A_1B_2 \cap s = \{C\}$ $A_1B_1 \cap s = \{D\}$



$\Delta ACA_1 \sim \Delta BCB_2 \Rightarrow$ $CA : CB = AA_1 : BB_2 = p : q$ $\Rightarrow$ $C \in$GMT

$\Delta ADA_1 \sim \Delta BDB_1 \Rightarrow$ $DA : DB = AA_1 : BB_1 = p : q \Rightarrow$ $D \in$GMT

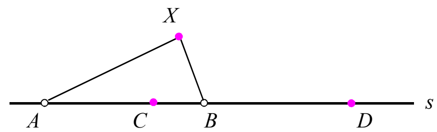
GMT $\cap$ $s = \{C, D\}$

2^o $X \notin s$

$XA : XB = p : q$

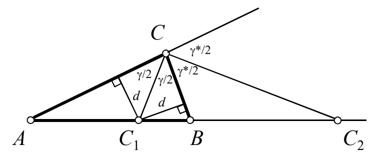
$CA : CB = DA : DB = p : q$

$XA : XB = CA : CB = DA : DB$



Lema 2. U ΔABC, gde je $AC > BC$, simetrale unutrašnjeg i spoljašnjeg ugla kod temena C seku pravu AB u tačkama C_1 i C_2, redom. Tada je $CA : CB = C_1A : C_1B = C_2A : C_2B$.

Dokaz.



$$\frac{P(CAC_1)}{P(CBC_1)} = \frac{\frac{1}{2}\,CA \cdot d}{\frac{1}{2}\,CB \cdot d} = \frac{CA}{CB} \qquad (1)$$

$$\frac{P(CAC_1)}{P(CBC_1)} = \frac{\frac{1}{2}\,C_1A \cdot h_c}{\frac{1}{2}\,C_1B \cdot h_c} = \frac{C_1A}{C_1B} \qquad (2)$$

(1), (2) $\Rightarrow$ $CA : CB = C_1A : C_1B$

slično iz $\frac{P(CAC_2)}{P(CBC_2)} \Rightarrow CA : CB = C_2A : C_2B$

■

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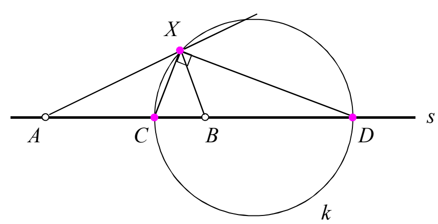
$XA : XB = CA : CB = DA : DB$

L 2 $\Rightarrow$ XC, XD – simetrale uglova kod X u ΔABX



$\Rightarrow \angle CXD = 90^0$

$X \in$ GMT – $\{C, D\}$ $\Rightarrow$ $\angle CXD = 90^0$

G = $k(CD)$ – prečnik CD

(1) $X \in$ GMT $\overset{1^o,\,2^o}{\Rightarrow}$ $X \in$ G

(2) $X \in$ G $\Rightarrow$ $X \in$ GMT

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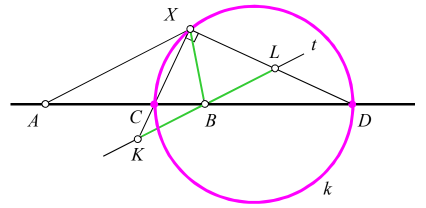
$X \in k(CD)$

$t(B) \parallel XA$

$t \cap XC = \{K\} \quad t \cap XD = \{L\}$

$\Delta XAC \sim \Delta BKC$

$$\frac{XA}{BK} = \frac{CA}{CB} = \frac{p}{q} \qquad (3)$$



$\Delta XAD \sim \Delta LBD$

$$\frac{XA}{BL} = \frac{DA}{BB} = \frac{p}{q} \qquad (4)$$

$$(3), (4) \Rightarrow \frac{XA}{BK} = \frac{XA}{BL} \Rightarrow BK = BL$$

$\angle KXL = \angle CXD = 90^0 \Rightarrow B$ – centar opisane kružnice oko ΔKXL

$$\Rightarrow BX = BK = BL \qquad (5)$$

$$\frac{XA}{XB} \overset{(5)}{=} \frac{XA}{BK} \overset{(3)}{=} \frac{p}{q} \Rightarrow X \in \text{GMT}$$

$1^o, 2^o \Rightarrow \text{GMT} = k$

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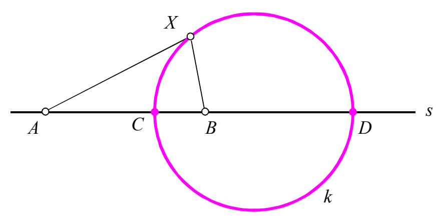
Napomena. k – *Apolonijeva kružnica* za tačke A i B i odnos $p : q$

r – poluprečnik k

$$r = \tfrac{1}{2}\, CD \qquad (1)$$



$$\frac{CA}{CB} = \frac{p}{q} \Rightarrow \frac{CA + CB}{CB} = \frac{p+q}{q}$$

$$\Rightarrow \frac{AB}{CB} = \frac{p+q}{q}$$

$$\Rightarrow CB = \frac{q \cdot AB}{p+q} \qquad (2)$$

$$\frac{DA}{DB} = \frac{p}{q} \Rightarrow \frac{DA - DB}{DB} = \frac{p-q}{q} \Rightarrow \frac{AB}{DB} = \frac{p-q}{q}$$

$$\Rightarrow DB = \frac{q \cdot AB}{p-q} \qquad (3)$$

$$CD = CB + DB \overset{(2),(3)}{=} \frac{q \cdot AB}{p+q} + \frac{q \cdot AB}{p-q} = \frac{2pq}{p^2 - q^2} AB \overset{(1)}{\Rightarrow} r = \frac{pq}{p^2 - q^2} AB$$

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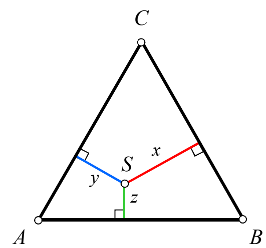
Primer 15. *U unutrašnjosti jednakostraničnog ΔABC odrediti GM tačaka S, takvih da se od duži koje su jednake rastojanjima tačke S od stranica ΔABC može konstruisati trougao.*

Rešenje. GMT $\neq \emptyset$

O – centar $k(A, B, C)$ $\Rightarrow$ $x = y = z = \frac{1}{3}h$

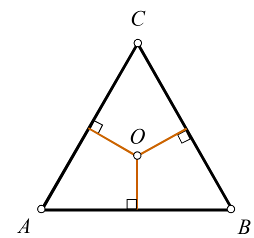
$O \in$ GMT



$S \in$ GMT $\Rightarrow$ $x < y + z$

$y < z + x$

$z < x + y$



GMT = ? G = ?

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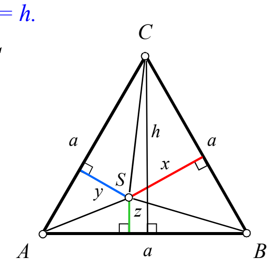
Lema 3. (Viviani, 1622-1703) *Neka je S proizvoljna tačka u unutrašnjosti jednakostraničnog ΔABC i neka su x, y, z njena rastojanja od stranica BC, CA, AB, redom. Ako je h visina ΔABC, tada je $x + y + z = h$.*

Dokaz. (I) $BC = CA = AB = a$ h – visina ΔABC

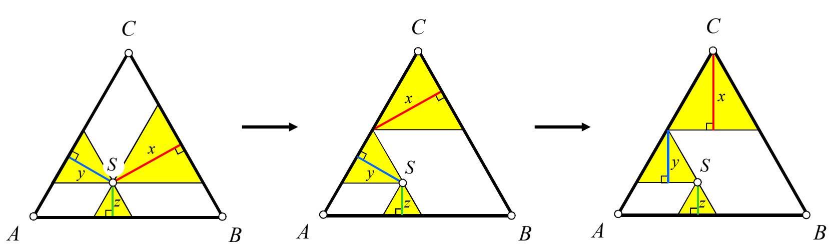
$P_{BCS} + P_{CAS} + P_{ABS} = P_{ABC}$

$\frac{1}{2}ax + \frac{1}{2}ay + \frac{1}{2}az = \frac{1}{2}ah$

$\Rightarrow x + y + z = h$



(II) "bez reči"



$x + y + z = h$ ■

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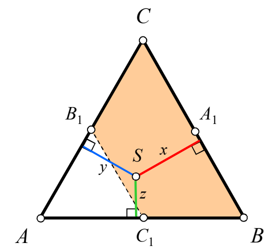
$x \in$ GMT

$x < y + z$ (1)

$y < z + x$ (2)

$z < x + y$ (3)

$x + y + z = h$ (4)

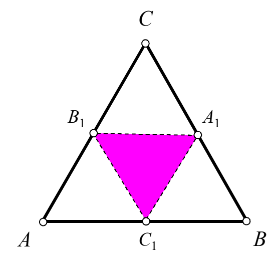


$(1) + (4) \Rightarrow 2x < h \Rightarrow x < \frac{h}{2}$

A_1, B_1, C_1 – sredine BC, CA, AB

$\Rightarrow S \in \text{int } B_1C_1BC$ (5)

slično



$y < \frac{h}{2} \Rightarrow S \in \text{int } C_1A_1CA$ (6)

$z < \frac{h}{2} \Rightarrow S \in \text{int } A_1B_1AB$ (7)

$(5), (6), (7) \Rightarrow S \in \text{int } B_1C_1BC \cap \text{int } C_1A_1CA \cap \text{int } A_1B_1AB = \text{int } A_1B_1C_1$

G = int $A_1B_1C_1$

G = GMT

■

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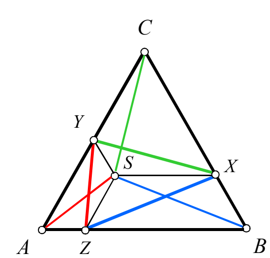
Primer 16. *U ravni jednakostraničnog ΔABC odrediti GM tačaka S, takvih da se od duži SA, SB, SC može konstruisati trougao.*

Rešenje. pretraga ravni ABC

1° $S \in \text{int } \Delta ABC \cup (\Delta ABC - \{A, B, C\})$

(I varijanta) konstruktivni dokaz

(a) $S \in \text{int } \Delta ABC$



$SX \parallel AB$, $X \in BC$

$SY \parallel BC$, $Y \in CA$

$SZ \parallel CA$, $Z \in AB$

$YAZS$, $ZBXS$, $XCYS$ – jednakokraki trapezi

$\Rightarrow SA = YZ$ $SB = ZX$ $SC = XY$

X, Y, Z nekolinearne

$\Rightarrow \Delta XYZ$ – traženi

$\Rightarrow \text{int } \Delta ABC \subset$ GMT

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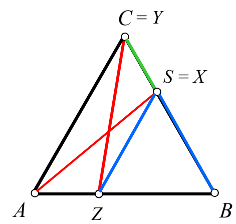
(b) $\Delta ABC - \{A, B, C\}$ $S \in BC$
$X = S$ $Y = C$

$SA = YZ$ $SB = ZX \equiv ZS$ $SC \equiv XY$

$\Rightarrow$ $\Delta XYZ \equiv \Delta SCZ$ – traženi

$\Rightarrow$ $\Delta ABC - \{A, B, C\} \subset$ GMT



(II varijanta) egzistencijalni dokaz

(a) $S \in \text{int}\, \Delta ABC$

$\Delta ABS \Rightarrow SA + SB > AB$ (1)

$\angle ADC \geq 90^0 \vee \angle BDC \geq 90^0$

$\Rightarrow AC > CD > SC$ (2)

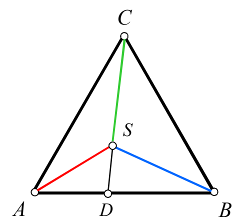
(1) , (2) $\Rightarrow SA + SB > SC$ (3)

slično $SB + SC > SA$ (4)

$SC + SA > SB$ (5)

(3), (4), (5) $\Rightarrow \exists\, \Delta$ sa stranicama SA, SB, SC

(b) $S \in \Delta ABC - \{A, B, C\}$ kao (a)



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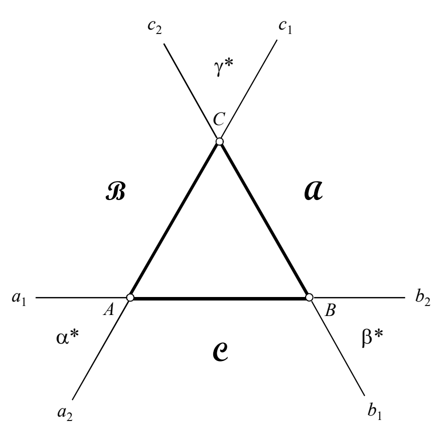
2^o $S \in \text{ext}\, \Delta ABC$

ext $\Delta ABC = \alpha^* \cup \beta^* \cup \gamma^* \cup a_1 \cup a_2 \cup b_1 \cup b_2 \cup c_1 \cup c_2 \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$



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(a) $S \in \gamma^*$

$SX \parallel AB \quad X \in c_2$

$SY \parallel BC \quad Y \in c_1$

$SZ \parallel CA \quad Z \in a_1$

$ZAYS$, $ZBSX$, $XCYS$ – jednakokraki trapezi

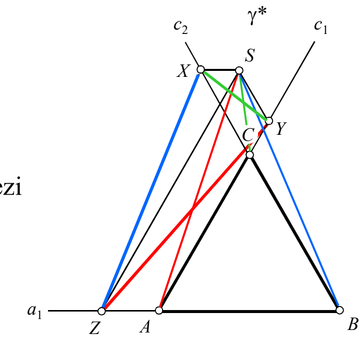
$\Rightarrow \; SA = YZ \qquad SB = ZX \qquad SC = XY$

X, Y, Z nekolinearne

$\Rightarrow \; \Delta XYZ$ – traženi

$\Rightarrow \; \gamma^* \subset$ GMT

slično $\alpha^*, \beta^* \subset$ GMT



(b) $S \in c_1$

$SX \parallel AB \quad X \in c_2$

$SY \parallel BC \quad Y \in c_1 \qquad Y = S$

$SZ \parallel CA \quad Z \in a_1 \qquad Z = A$

$SA \equiv YZ$

$SB = ZX$ ($ZBSX$ – jednakokraki trapez)

$SC = XS \equiv XY$ (ΔCSX – jednakostranični)

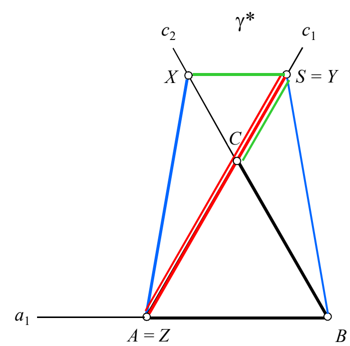
X, Y, Z nekolinearne

$\Rightarrow \; \Delta XYZ \equiv \Delta XSA$ – traženi

$\Rightarrow \; c_1 - \{C\} \subset$ GMT

slično

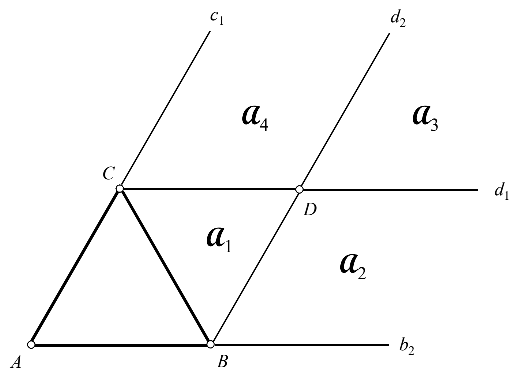
$c_2 - \{C\}, a_1 - \{A\}, a_2 - \{A\}, b_1 - \{B\}, b_2 - \{B\} \subset$ GMT



(c) $S \in \alpha$

ΔBDC – jednakostranični

$\alpha = \underline{\alpha_1} \cup \underline{\alpha_2} \cup \underline{\alpha_3} \cup \alpha_4 \cup \underline{(BD]} \cup (CD] \cup \underline{d_1} \cup d_2$



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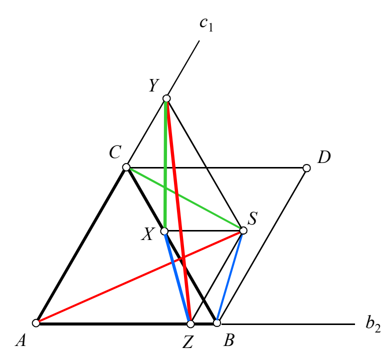
(c1) $S \in \alpha_1 \Leftrightarrow S \in \text{int}\, \Delta BDC$

$SX \parallel AB$, $X \in BC$
$SY \parallel BC$, $Y \in c_1$
$SZ \parallel CA$, $Z \in AB$

$SA = YZ$ ($YAZS$ – jednakokraki trapez)
$SB = ZX$ ($ZBSX$ – jednakokraki trapez)
$SC = XY$ ($SYZX$ – jednakokraki trapez)

ako su X, Y, Z nekolinearne
$\Rightarrow \Delta XYZ$ – traženi



$\alpha_1 \subset$ GMT ?

Šta ako su tačke X, Y, Z kolinearne?
Da li to može da se dogodi?
Ako može, kada?

42

Lema 4. *Neka* $S \in \mathcal{A}_1$. *Tačke X, Y, Z su kolinearne ako i samo ako* $S \in k(A, B, C)$.

Dokaz. ($\Rightarrow$) X, Y, Z kolinearne

$\angle ZXB = \angle YXC = \theta$ (unakrsni) (1)

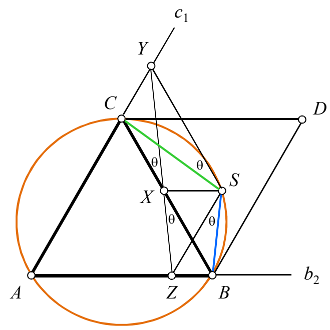
$ZBSX$ – jednakokraki trapez $\Rightarrow$ tetivni četv.

$\Rightarrow \angle ZXB = \angle ZSB$ (2)

$SYXC$ – jednakokraki trapez $\Rightarrow$ tetivni četv.

$\Rightarrow \angle YXC = \angle YSC$ (3)

(1), (2), (3) $\Rightarrow \angle ZSB = \angle YSC = \theta$ (4)



$YAZS$ – jednakokraki trapez $\Rightarrow \angle ZSY = 120^0$ (5)

(4), (5) $\Rightarrow \angle BSC = \angle BSY - \angle YSC = \angle BSY - \theta = \angle BSY - \angle ZSB$

$= \angle ZSY = 120^0$ (6)

(6), $\angle CAB = 60^0 \Rightarrow ABSC$ tetivni $\Rightarrow S \in k(A, B, C)$

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($\Leftarrow$) $S \in k(A, B, C)$

$S \in \mathcal{A}_1 \Rightarrow S \in$ luk $BC \Rightarrow \angle BSC = 120^0$ (7)

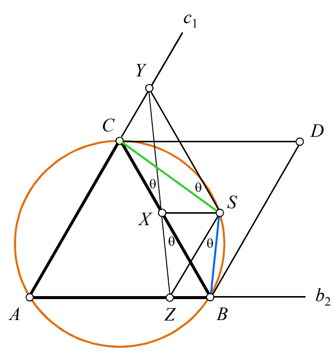
$\angle ZSY = 120^0 \overset{(7)}{\Rightarrow} \angle ZSB = \angle YSC = \theta$ (8)

$\angle ZXB = \angle ZSB$ ($ZBSX$ - tetivni četv.) (9)

$\angle YXC = \angle YSC$ ($SYXZ$ - tetivni četv.) (10)

(8), (9), (10) $\Rightarrow \angle ZXB = \angle YXC = \theta$

$\Rightarrow X, Y, Z$ kolinearne



■

L 3 $\Rightarrow \mathcal{A}_1$ – luk $BC \subset$ GMT

slično

$\mathcal{B}_1$ – luk BC, $\mathcal{C}_1$ – luk $BC \subset$ GMT

44

(c2) $S \in \mathcal{A}_2$

$SX \parallel AB \quad X \in BC$

$SY \parallel BC \quad Y \in c_1$

$SZ \parallel CA \quad Z \in b_2$

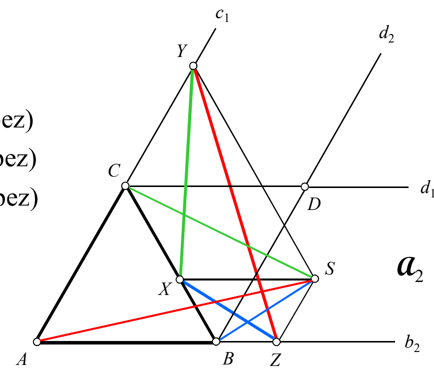
$SA = YZ$ ($YAZS$ – jednakokraki trapez)

$SB = ZX$ ($BZSX$ – jednakokraki trapez)

$SC = XY$ ($SYCX$ – jednakokraki trapez)

X, Y, Z nekolinearne

$\Rightarrow$ ΔXYZ – traženi



$\Rightarrow$ $\mathcal{A}_2 \subset$ GMT

slično

$\mathcal{A}_4$, $\mathcal{B}_2$, $\mathcal{B}_4$, $\mathcal{C}_2$, $\mathcal{C}_4 \subset$ GMT

45

(c3) $S \in \mathcal{A}_3$

$SX \parallel AB \quad X \in c_2$

$SY \parallel BC \quad Y \in c_1$

$SZ \parallel CA \quad Z \in b_2$

$SA = YZ$ ($YAZS$ – jednakokraki trapez)

$SB = ZX$ ($BZSX$ – jednakokraki trapez)

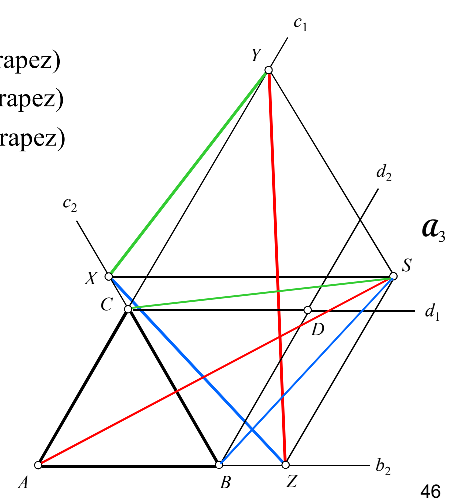
$SC = XY$ ($SYXC$ – jednakokraki trapez)

X, Y, Z nekolinearne

$\Rightarrow$ ΔXYZ – traženi

$\Rightarrow$ $\mathcal{A}_3 \subset$ GMT

slično, $\mathcal{B}_3$, $\mathcal{C}_3 \subset$ GMT



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(c4) $S\in(BD]$

$SX \parallel AB \quad X\in BC$

$SY \parallel BC \quad Y\in c_1$

$SZ \parallel CA \quad Z = B$

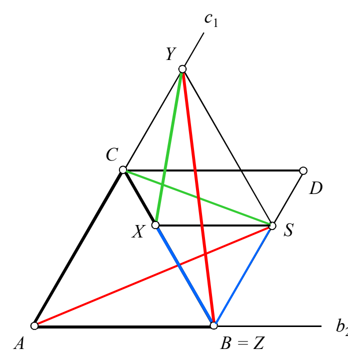
$SA = YZ \equiv YB$ ($YABZ$ – jednakokraki trapez)

$SB \equiv SZ = ZX$ (ΔBSX – jednakostranični)

$SC = XY$ ($YABS$ – jednakokraki trapez)

X, Y, Z nekolinearne

$\Rightarrow \Delta XYZ \equiv \Delta XZB$ – traženi



$S = D \Rightarrow X = C,\ Y\in c_1,\ Z = B$

$\Rightarrow \Delta XYZ \equiv \Delta CYB$ – traženi

$\Rightarrow (BD] \subset$ GMT

slično, $(CD]$ i odgovarajuće duži u $\mathcal{B}$ i $\mathcal{C} \subset$ GMT

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(c5) $S\in d_1$

$SX \parallel AB \quad X = C$

$SY \parallel BC \quad Y\in c_1$

$SZ \parallel CA \quad Z\in b_2$

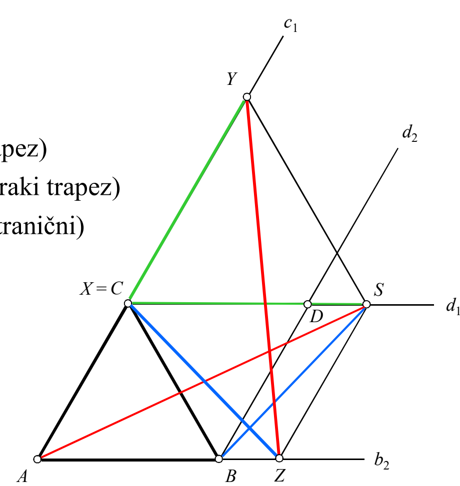
$SA = YZ$ ($YAZS$ – jednakokraki trapez)

$SB = ZX \equiv ZC$ ($BZSC$ – jednakokraki trapez)

$SC \equiv SX = XY$ (ΔCSY – jednakostranični)

X, Y, Z nekolinearne

$\Rightarrow \Delta XYZ \equiv \Delta XZB$ – traženi



$\Rightarrow d_1 \subset$ GMT

slično, d_2 i odgovarajuće duži u $\mathcal{B}$ i $\mathcal{C} \subset$ GMT

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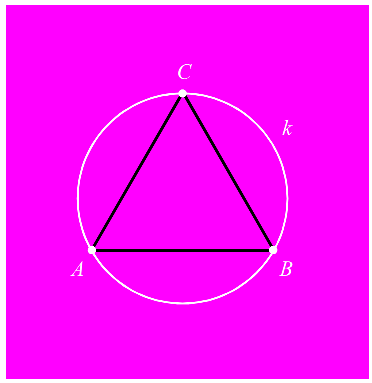
$1^\circ, 2^\circ \;\Rightarrow\;$ GMT $= \text{int}\,\Delta ABC \cup (\Delta ABC - \{A, B, C\})$

$\cup\, (\mathcal{A} - \text{luk } BC) \cup (\mathcal{B} - \text{luk } CA) \cup (\mathcal{C} - \text{luk } AB)$

$= r(A, B, C) - k(A, B, C)$



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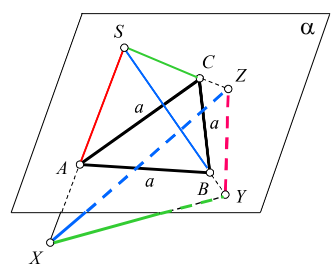
Primer 17. *U prostoru je dat jednakostranični ΔABC. Odrediti GM tačaka S, takvih da se od duži SA, SB, SC može konstruisati trougao.*

Rešenje. $r(A, B, C) = \alpha \quad BC = CA = AB = a$

$1^\circ\; S \in \alpha \quad$ GMT $= r(A, B, C) - k(A, B, C) \quad$ (primer 16)

$2^\circ\; S \notin \alpha$



$X \in SA\,,\; SX = \dfrac{SB \cdot SC}{a} \qquad (1)$

$Y \in SB\,,\; SY = \dfrac{SC \cdot SA}{a} \qquad (2)$

$Z \in SC\,,\; SY = \dfrac{SA \cdot SB}{a}$

$(1), (2) \;\Rightarrow\; SA \cdot SX = SB \cdot SY = \dfrac{SA \cdot SB \cdot SC}{a}$

$\Rightarrow\; \dfrac{SX}{SY} = \dfrac{SB}{SA} \;\Rightarrow\; \Delta XYS \sim \Delta BAS$

$\Rightarrow\; \dfrac{XY}{AB} = \dfrac{SX}{SB} \;\Rightarrow\; XY = AB \cdot \dfrac{SX}{SB} \overset{(1)}{=} a \cdot \dfrac{\frac{SB \cdot SC}{a}}{SB} = SC$

$XY = SC$

slično, $YZ = SA\,,\; YX = SB$

$\Rightarrow\; \Delta XYZ$ – traženi $\qquad \mathcal{P}$ – prostor $\;\Rightarrow\;$ GMT $= \mathcal{P} - k(A, B, C)$

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